GEOMETRIC VALUATIONS

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Introduction

This paper is the result of a National Science Foundation-funded Research Experience for Undergraduates (REU) at the University of Notre Dame during the summer of 2006. The REU was directed by Professor Frank Connolly, and the research project was supervised by Professor Liviu Nicolaescu.

The topic of our independent research project for this REU was Geometric Probability. Consequently, the first half of this paper is a study of the book *Introduction to Geometric Probability* by Daniel Klain and Gian-Carlo Rota. While closely following the text, we have attempted to clarify and streamline the presentation and ideas contained therein. In particular, we highlight and emphasize the key role played by the Radon transform in the classification of valuations. In the second part we take a closer look at the special case of valuations on polyhedra.

Our primary focus in this project is a type of function called a "valuation". A valuation associates a number to each "reasonable" subset of $\mathbb{R}^n$ so that the inclusion-exclusion principle is satisfied.

Examples of such functions include the Euclidean volume and the Euler characteristic. Since the objects we are most interested lie in an Euclidean space and moreover we are interested in properties which are independent of the location of the objects in space, we are motivated to study "invariant" valuations on certain subsets of $\mathbb{R}^n$. The goal of the first half of this paper, therefore, is to characterize all such valuations for certain natural collections of subsets in $\mathbb{R}^n$.

This undertaking turned out to be quite complex. We must first spend time introducing the abstract machinery of valuations (chapter 1), and then applying this machinery to the simpler simpler case of pixelations (chapter 2). Chapter 3 then sets up the language of polyconvex sets, and explains how to use the Radon transform to generate many new examples of valuations. These new valuations have probabilistic interpretations. In chapter 4 we finally nail the characterization of invariant valuations on polyconvex sets. Namely, we show that all valuations are obtainable by the Radon transform technique from a unique valuation, the Euler characteristic. We celebrate this achievement in chapter 5 by exploring applications of the theory.

With the valuations having been completely characterized, we turn our attention toward special polyconvex sets: polyhedra, that is finite unions of *convex* polyhedra. These polyhedra can be triangulated, and in chapter 5 we investigate the combinatorial features of a triangulation, or simplicial complex.

In Chapter 7 we prove a global version of the inclusion-exclusion principle for simplicial complexes known as the Möbius inversion formula. Armed we this result, we then explain how to compute the valuations of a polyhedron using data coming form a triangulation.

In Chapter 8 we use the technique of integration with respect to the Euler characteristic to produce combinatorial versions of Morse theory and Gauss-Bonnet formula. In the end, we arrive at an explicit formula relating the Euler characteristic of a polyhedron in terms of measurements taken at vertices. These measurements can be interpreted either as curvatures, or as certain averages of Morse indices.

The preceding list says little about why one would be interested in these topics in the first place. Consider a coffee cup and a donut. Now, a topologist would you tell you that these two items are “more or less the same.” But that’s ridiculous! How many times have you eaten a coffee cup? Seriously, even aside from the fact that they are made of different materials, you have to admit that there is a geometric difference between a coffee cup and a donut. But what? More generally, consider the shapes around you. What is it that distinguishes them from each other, geometrically?

We know certain functions, such as the Euler characteristic and the volume, tell part of the story. These functions share a number of extremely useful properties such as the inclusion-exclusion principle, invariance, and “continuity”. This motivates us to consider all such functions. But what are all such functions? In order to apply these tools to study polyconvex sets, we must first understand the tools at our disposal. Our attempts described in the previous paragraphs result in a full characterization of valuations on polyconvex sets, and even lead us to a number of useful and interesting formulae for computing these numbers.

We hope that our efforts to these ends adequately communicate this subject’s richness, which has been revealed to us by our research advisor Liviu Nicolaescu. We would like to thank him for his enthusiasm in working with us. We would also like to thank Professor Connolly for his dedication in directing the Notre Dame REU and the National Science Foundation for supporting undergraduate research.

CONTENTS

1. **Valuations on lattices of sets**

§1.1. **Valuations.**

Definition 1.1. (a) For every set S we denote by $P(S)$ the collection of subsets of S and by $\mathrm{Map}(S, \mathbb{Z})$ the set of functions $f : S \to \mathbb{Z}$. The *indicator* or *characteristic* function of a subset $A \subset S$ is the function $I_A \in \mathrm{Map}(S, \mathbb{Z})$ defined by

$$I_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$

(b) If S is a set then an S-*lattice* (or a lattice of sets) is a collection $\mathcal{L} \subset P(S)$ such that

$$\emptyset \in \mathcal{L}, \quad A, B \in \mathcal{L} \Longrightarrow A \cap B, \quad A \cup B \in \mathcal{L}.$$

(c) Is $\mathcal{L}$ is an S-lattice then a subset $\mathcal{G} \in \mathcal{L}$ is called *generating* if

$$\emptyset \in \mathcal{G} \ \text{ and } \ A, B \in \mathcal{G} \Longrightarrow A \cap B \in G,$$

and every $A \in \mathcal{L}$ is a finite union of sets in $\mathcal{G}$. □

Definition 1.2. Let G be an Abelian group and S a set.
(a) A G-*valuation* on an S-lattice of sets is a function $\mu : \mathcal{L} \to G$ satisfying the following conditions:
  (a1) $\mu(\emptyset) = 0$
  (a2) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$. (Inclusion-Exclusion)
(b) If $\mathcal{G}$ is a generating set of the S-lattice $\mathcal{L}$, then a G-valuation on $\mathcal{G}$ is a function $\mu : \mathcal{G} \to G$ satisfying the following conditions:
  (b1) $\mu(\emptyset) = 0$
  (b2) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, for every $A, B \in \mathcal{G}$ such that $A \cup B \in \mathcal{G}$. □

The inclusion-exclusion identity in Definition 1.2 implies the *generalized inclusion-exclusion identity*

$$\mu(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) + \cdots \tag{1.1}$$

Example 1.3. (a) (*The universal valuation*) Suppose S is a set. Observe that $\mathrm{Map}(S, \mathbb{Z})$ is a commutative ring with 1. The map

$$I_\bullet : P(S) \to \mathrm{Map}(S, \mathbb{Z})$$

given by

$$P(S) \ni A \mapsto I_A \in \mathrm{Map}(S, \mathbb{Z})$$

is a valuation. This follows from the fact that the indicator functions satisfy the following identities.

$$I_{A \cap B} = I_A I_B \tag{1.2a}$$

$$I_{A \cup B} = I_A + I_B - I_{A \cap B} = I_A + I_B - I_A I_B = 1 - (1 - I_A)(1 - I_B). \tag{1.2b}$$

(b) Suppose S is a finite set. Then the *cardinality* map

$$\# : P(S) \to \mathbb{Z}, \quad A \mapsto \#A := \text{the cardinality of } A$$

is a valuation.
(c) Suppose $S = \mathbb{R}^2$, $R = \mathbb{R}$ and $\mathcal{L}$ consists of measurable bounded subsets of the Euclidean space. The map which associates to each $A \in \mathcal{L}$ its Euclidean area, $\mathrm{area}(A)$, is a real valued valuation. Note that the lattice point count map

$$\lambda : \mathcal{L} \to \mathbb{Z}, \quad A \mapsto \#(A \cap \mathbb{Z}^2)$$

is a $\mathbb{Z}$-valuation.
(d) Let S and $\mathcal{L}$ be as above. Then the Euler characteristic defines a valuation

$$\chi : \mathcal{L} \to \mathbb{Z}, \quad A \mapsto \chi(A). \qquad \square$$

If $(G, +)$ is an Abelian group and R is a commutative ring with 1 then we denote by $\mathrm{Hom}_{\mathbb{Z}}(G, R)$ the set of group morphisms $G \to R$, i.e. the set of maps

$$\varphi : G \to R$$

such that

$$\varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2), \quad \forall g_1, g_2 \in G.$$

We will refer to the maps in $\mathrm{Hom}_{\mathbb{Z}}(G, R)$ as $\mathbb{Z}$-linear maps from G to R.
Suppose $\mathcal{L}$ is an S-lattice. We denote by $\mathcal{S}(\mathcal{L})$ the (additive) subgroup of $\mathrm{Map}(S, \mathbb{Z})$ generated by the functions I_A, $A \in \mathcal{L}$. We will refer to the functions in $\mathcal{S}(\mathcal{L})$ as $\mathcal{L}$-*simple functions*, or simple functions if the lattice $\mathcal{L}$ is clear from the context.

Definition 1.4. Suppose $\mathcal{L}$ is an S-lattice, and G is an Abelian group. An G-*valued integral* on $\mathcal{L}$ is a $\mathbb{Z}$-linear map

$$\int : \mathcal{S}(\mathcal{L}) \to G, \quad \mathcal{S}(\mathcal{L}) \ni f \longmapsto \int f \in G. \qquad \square$$

Observe that any G-valued integral on an S-lattice $\mathcal{L}$ defines a valuation $\mu : \mathcal{L} \to G$ by setting

$$\mu(A) := \int I_A.$$

The inclusion-exclusion formula for μ follows from (1.2a) and (1.2b). We say that μ is the *valuation induced by the integral*. When a valuation is induced by an integral we will say that the *valuation induces an integral*.
In general, a generating set of a lattice has a much simpler structure and it is possible to construct many valuations on it. A natural question arises: Is it possible to extend to the entire lattice a valuation defined on a generating set? The next result describes necessary and sufficient conditions for which this happens.

§1.2. **Extending Valuations.**

Theorem 1.5 (Groemer's Integral Theorem)**.** *Let $\mathcal{G}$ be a generating set for a lattice $\mathcal{L}$ and let $\mu : \mathcal{G} \to H$ be a valuation on $\mathcal{G}$, where H is an Abelian group. The following statements are equivalent.*

(1) *μ extends uniquely to a valuation on $\mathcal{L}$.*
(2) *μ satisfies the inclusion-exclusion identities*

$$\mu(B_1 \cup B_2 \cup \cdots \cup B_n) = \sum_i \mu(B_i) - \sum_{i<j} \mu(B_i \cap B_j) + \cdots$$

for every $n \geq 2$ and any $B_i \in \mathcal{G}$ such that $B_1 \cup B_2 \cup \cdots \cup B_n \in \mathcal{G}$.
(3) *μ induces an integral on the space of simple functions $\mathcal{S}(\mathcal{L})$.*

Proof. We follow closely the presentation in [**KR**, Chap.2].

• $(1) \implies (2)$. Note that the second statement is not trivial because $B_1 \cup \cdots \cup B_{n-1}$ is not necessarily in $\mathcal{G}$. Suppose the valuation μ extends uniquely to a valuation on L. Then μ satisfies the inclusion exclusion identity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{L}$. Since $B_1 \cup \cdots \cup B_{n-1} \in \mathcal{L}$ even if it is not in $\mathcal{G}$, we can apply the inclusion-exclusion identity repeatedly to obtain the result.

• $(2) \implies (3)$. We wish to construct a linear map $\int : \mathcal{S}(\mathcal{L}) \to H$. To do this, we first note that by (1.2b) any function, f in $\mathcal{S}(\mathcal{L})$ can be written as

$$f = \sum_{i=1}^{m} \alpha_i I_{K_i},$$

where $K_i \in \mathcal{G}$ and $\alpha_i \in \mathbb{Z}$. We thus define an integral as follows:

$$\int \sum_{i=1}^{m} \alpha_i I_{K_i} d\mu := \sum_{i=1}^{m} \alpha_i \mu(K_i).$$

This map might not be well-defined since f could be represented in different ways as a linear combination of indicator functions of generating sets. We thus need to show that the above map is independent of such a representation. We argue by contradiction and we assume that f has two distinct representations

$$f = \sum \gamma_i I_{A_i} = \sum \beta_i I_{B_i},$$

yet

$$\sum \gamma_i \mu(A_i) \neq \sum \beta_i \mu(B_i).$$

Thus, subtracting these equations and renaming the terms appropriately, we are left with the situation

$$\sum_{i=1}^{m} \alpha_i I_{K_i} = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \mu(K_i) \neq 0. \tag{1.3}$$

Now we label the intersections

$$L_1 = K_1, \quad \ldots, \quad L_m = K_m, \quad L_{m+1} = K_1 \cap K_2, \quad L_{m+2} = K_1 \cap K_3, \ldots$$

such that $L_i \subset L_j \Rightarrow j < i$. This can be done because we have a finite number of sets. We note that all the L_i's are in $\mathcal{G}$, since the K_i's and their intersections are in $\mathcal{G}$. We then rewrite (1.3) in terms of the L_i's as

$$\sum_{i=1}^{p} a_i I_{L_i} = 0 \quad \text{and} \quad \sum_{i=1}^{p} a_i \mu(L_i) \neq 0. \tag{1.4}$$

Now take q maximal such that

$$\sum_{i=q}^{p} a_i I_{L_i} = 0 \quad \text{and} \quad \sum_{i=q}^{p} a_i \mu(L_i) \neq 0. \tag{1.5}$$

Note that $1 \leq q < p$. Note that $a_q \neq 0$ since then q would not be maximal.

Let us now observe that

$$L_q \subset \bigcup_{i=q+1}^{p} L_i.$$

Indeed, if $x \in L_q \setminus \bigcup_{j=q+1}^{p} L_j$ then

$$I_{L_i}(x) = 0 \ \ \forall i \neq q \ \text{ and } \ a_q = \sum_{i=q}^{p} a_i I_{L_i}(x) = 0.$$

This is impossible since $a_q \neq 0$. Hence

$$L_q = L_q \cap \left(\bigcup_{i=q+1}^{p} L_i \right) = \bigcup_{i=q+1}^{p} (L_q \cap L_i).$$

Let us write $L_q \cap L_i = L_{j_i}$. Then, since $i > q$ and by construction $L_i \subset L_j \implies j < i$, we have that $j_i > q$. Thus:

$$L_q = \bigcup_{i=q+1}^{p} (L_q \cap L_i) = \bigcup_{i=q+1}^{p} L_{j_i}.$$

Then we have

$$0 \neq \sum_{i=q}^{n} a_i \mu(L_i) = a_q \mu(L_q) + \sum_{i=q+1}^{n} a_i \mu(L_i)$$
$$= a_q \mu \left(\bigcup_{i=q+1}^{p} L_{j_i} \right) + \sum_{i=q+1}^{n} a_i \mu(L_i) = \sum_{i=q+1}^{p} b_i \mu L_i,$$

where the last equality is attained by applying the assumed inclusion/exclusion principle to the union and regrouping the terms.

We now repeat exactly the same process with the expression involving the indicator function. Then,

$$\sum_{i=q}^{p} a_i I_{L_i} = a_q I_{\bigcup_{i=q+1}^{p}(L_q \cap L_i)} + \sum_{i=q+1}^{p} a_i I_{L_i} = \sum_{i=q+1}^{p} b_i I_{L_i}$$

Thus,

$$\sum_{i=q+1}^{p} b_i I_{L_i} = 0$$

However, this contradicts the maximality of q. Hence, the integral map is well defined, and we are done.

• $(3) \implies (1)$. Suppose μ defines an integral on the space of $\mathcal{L}$-simple functions. Then for $A \in G$, $\int I_A d\mu = \mu(A)$. This motivates us to define an extension $\tilde{\mu}$ of μ to $\mathcal{L}$ by

$$\tilde{\mu}(A) := \int I_A d\mu = \mu(A), \quad A \in \mathcal{L}.$$

This definition is certainly unambiguous and its restriction to $\mathcal{G}$ is just μ, so we need only check that it is a valuation. Let $A, B \in \mathcal{L}$. Then,

$$\tilde{\mu}(A \cup B) = \int I_{A\cup B} d\mu = \int I_A + I_B - I_{A\cap B} d\mu$$
$$= \int I_A d\mu + \int I_B d\mu - \int I_{A\cap B} d\mu = \tilde{\mu}(A) + \tilde{\mu}(B) - \tilde{\mu}(A \cap B).$$

Thus, $\tilde{\mu}$ is an extension of μ to $\mathcal{L}$. Moreover, it is unique.

Suppose ν is another extension of μ. Then, given any $A \in L$ we can write $A = K_1 \cup \ldots \cup K_r$ for $K_i \in G$. Since $\tilde{\mu}$ and ν are both valuations, both satisfy the generalized inclusion exclusion principle. Furthermore, since both are extensions of μ, both agree on $\mathcal{G}$

$$\tilde{\mu}(A) = \tilde{\mu}(K_1 \cup \ldots \cup K_r)$$
$$= \sum_{i=1}^{r} \mu(K_i) - \sum_{i<j} \mu(K_i \cap K_j) + \sum_{i<j<k} \mu(K_i \cap K_j \cap K_k) + \cdots$$
$$= \nu(K_1 \cup \ldots \cup K_r) = \nu(A).$$

Hence, the extension is unique. □

2. Valuations on pixelations

§2.1. **Pixelations.** We now study valuations on a small class of easy to understand subsets of $\mathbb{R}^n$. We will then use this knowledge to study valuations on much more general subsets of $\mathbb{R}^n$. One of our main aims is to classify all the "nice" valuations on such subsets. It will turn out that the ideas and results of this section are analogous to results discussed later.

Definition 2.1. (a) An *affine* k*-plane* in $\mathbb{R}^n$ is the translate of a k-dimensional vector subspace of $\mathbb{R}^n$. A *hyperplane* in $\mathbb{R}^n$ is an affine $(n-1)$-plane.
(b) An *affine transformation* of $\mathbb{R}^n$ is a *bijection* $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(\lambda\vec{u} + (1-\lambda)\vec{v}) = \lambda T\vec{u} + (1-\lambda)T\vec{v}, \quad \forall \lambda \in \mathbb{R}, \quad \vec{u}, \vec{v} \in \mathbb{R}^n.$$

The set $\mathrm{Aff}(\mathbb{R}^n)$ of affine transformations of $\mathbb{R}^n$ is a group with respect to the composition of maps. □

Definition 2.2. An (orthogonal) *parallelotope* in $\mathbb{R}^n$ a compact set P of the form

$$P = [a_1, b_1] \times \cdots [a_n, b_n], \quad a_i \leq b_i, \quad i = 1, \cdots n. \qquad \square$$

We will often refer to an orthogonal parallelotope as a parallelotope or even just a *box*.

Remark 2.3. It is entirely possible for any number of the intervals defining an orthogonal parallelotope to have length zero. Finally, note that the intersections of two parallelotopes is again a parallelotope. □

We denote by $\mathrm{Par}(n)$ the collection of parallelotopes in $\mathbb{R}^n$ and we define a *pixelation* to be a finite union of parallelotopes. Observe that the collection $\mathrm{Pix}(n)$ of pixelations in $\mathbb{R}^n$ is a lattice, i.e. it is stable under finite intersections and unions.

Definition 2.4. (a) A pixelation $P \in \mathrm{Pix}(n)$ is said to have *dimension* n (or *full dimension*) if P is *not* contained in a finite union of hyperplanes.
(b) A pixelation $P \in \mathrm{Pix}(n)$ is said to have *dimension* k ($k \leq n$) if P is contained in a finite union of k-planes but not in a finite union of $(k-1)$-planes. □

The top part of Figure 1 depicts possible boxes in $\mathbb{R}^2$, while the bottom part depicts a possible pixelation in $\mathbb{R}^2$.

§2.2. **Extending Valuations from Par to Pix.** By definition, $\mathrm{Par}(n)$ is a generating set of $\mathrm{Pix}(n)$. Consequently, we would like to know if we can extend valuations from $\mathrm{Par}(n)$ to $\mathrm{Pix}(n)$. The following theorem shows that we can do so whenever the valuations map into a commutative ring with 1.

Theorem 2.5. *Let R be a commutative ring with 1. Then any valuation $\mu : \mathrm{Par}(n) \to R$ extends uniquely to a valuation on $\mathrm{Pix}(n)$.*

Proof. Due to Groemer's Integral Theorem, all we need to show is that μ gives rise to an integral on the vector space of functions generated by the indicator functions of boxes. Thus

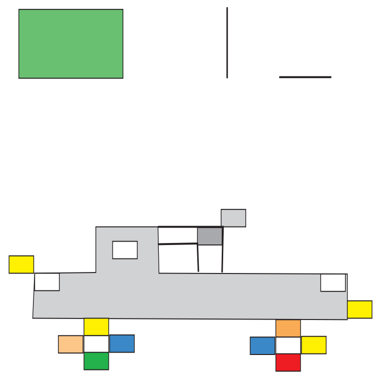

FIGURE 1. *Planar pixelations.*

it suffices to show that

$$\sum_{i=1}^{m} \alpha_i I_{P_i} = 0 \Longrightarrow \sum_{i=1}^{m} \alpha_i \mu(P_i) = 0,$$

where the P_i's are boxes.

We proceed by induction on the dimension n. If the dimension is zero, the space only has one point, and the above claim is true.

Now suppose that the theorem holds in dimension $n-1$. For the sake of contradiction, we suppose the theorem does not hold for dimension n. That is, suppose there exist distinct boxes $P_1, \ldots, P_m$ such that

$$\sum_{i=1}^{m} \alpha_i I_{P_i} = 0 \text{ and } \sum_{i=1}^{m} \alpha_i \mu(P_i) = r \neq 0. \tag{2.1}$$

Let k be the number of the boxes P_i of full dimension. Take k to be *minimal* over all such contradictions. We distinguish three cases.

Case 1. $k = 0$. Since none of the boxes is of full dimension, each is contained in a hyperplane. Of all the relations of type (2.1) we choose the one so that the boxes involved are contained in the smallest possible number ℓ of hyperplanes.

Assume first that $\ell = 1$. Then all the P_i are contained in a single hyperplane. By the induction hypothesis, then the integral is well defined, so we have a contradiction.

Thus, we can assume that $\ell > 1$. So, there exist ℓ hyperplanes orthogonal to the coordinate axes, $H_1, \ldots, H_\ell$, such that each P_i is contained in one of them. Without loss of generality, we may renumber the indices so that $P_1 \subset H_1$.

The the restriction to H_1 of the first sum in (2.1) is zero so that

$$\sum_{i=1}^{m} \alpha_i I_{P_i \cap H_1} = 0.$$

But, $P_i \cap H_1$ is a subset of the a hyperplane H_1 and we can apply the induction hypothesis to conclude that

$$\sum_{i=1}^{m} \alpha_i \mu(P_i \cap H_1) = 0.$$

Subtracting the above two equations from (2.1) we see that

$$\sum_{i=1}^{m}\alpha_i(I_{P_i}-I_{P_i\cap H_1})=0 \quad \text{and} \quad \sum_{i=2}^{m}\alpha_i(\mu(P_i)-\mu(P_i\cap H_1))=r\neq 0.$$

The above sums take the same form (2.1), but we see that the boxes $P_j\subset H_1$ disappear since $P_j=P_j\cap H_1$. Thus we obtain new equalities of the type (2.1) but the boxes involved are contained in fewer hyperplanes, $H_2,\cdots,H_\ell$ contradicting the minimality of ℓ.

Case 2. $k=1$. We may assume the top dimensional box is P_1. Then $P_2\cup\cdots\cup P_m$ is contained in a finite union of hyperplanes $H_1,\cdots,H_\nu$ perpendicular to the coordinate axes. Observe that

$$(H_1\cup\cdots\cup H_\nu)\cap P_1\subsetneq P_1.$$

Indeed,

$$\operatorname{vol}\big((H_1\cup\cdots\cup H_\nu)\cap P_1\big)\leq\sum_{j=1}^{\nu}\operatorname{vol}(H_j\cap P_1)=0<\operatorname{vol}(P_1),$$

so that

$$\exists x_0\in P_1\setminus(H_1\cup\cdots\cup H_\nu).$$

Using the identity $\sum\alpha_iI_{P_i}=0$ at x_0 found above we deduce $\alpha_1=0$ which contradicts the minimality of k.

Case 3. $k>1$. We can assume that the top dimensional boxes are $P_1,\cdots,P_k$.

Choose a hyperplane H such that $P_1\cap H$ is a facet of P_1, i.e. a face of P_1 of highest dimension such that it is not all of P_1. H has two associated closed half-spaces H^+ and H^-. H^+ is singled out by the requirement $P_1\subset H^+$. Recall that

$$\sum_{i=1}^{m}\alpha_iI_{P_i}=0.$$

Restricting to H^+ we deduce

$$\sum_{i=1}^{m}\alpha_iI_{P_i\cap H^+}=0.$$

Likewise,

$$\sum_{i=1}^{m}\alpha_iI_{P_i\cap H}=0 \quad \text{and} \quad \sum_{i=1}^{m}\alpha_iI_{P_i\cap H^-}=0 \tag{2.2}$$

Note that $P_i=(P_i\cap H^+)\cup(P_i\cap H^-)$ and $(P_i\cap H^+)\cap(P_i\cap H^-)=P_i\cap H$. Then, since μ is a valuation, it obeys the inclusion-exclusion rule so

$$\sum_{i=1}^{m}\alpha_i\mu(P_i)=\sum_{i=1}^{m}\alpha_i\mu(P_i\cap H^+)+\sum_{i=1}^{m}\alpha_i\mu(P_i\cap H^-)-\sum_{i=1}^{m}\alpha_i\mu(P_i\cap H) \tag{2.3}$$

Since the sets $P_i\cap H$ are in a space of dimension $n-1$, and

$$\sum_{i=1}^{m}\alpha_iI_{P_i\cap H}=0$$

we deduce from the induction assumption that

$$\sum_{i=1}^{m} \alpha_i \mu(P_i \cap H) = 0.$$

On the other hand, $P_1 \cap H^- = P_1 \cap H$ so $P_1 \cap H^-$ has dimension $n-1$. We now have a new collection of boxes $P_i \cap H^-$, $i = 1, \cdots, m$ of which at most $k-1$ are top dimensional and satisfy

$$\sum_{i} \alpha_i I_{P_i \cap H^-} = 0.$$

The minimality of k now implies

$$\sum_{i=1}^{m} \alpha_i \mu(P_i \cap H^-) = 0.$$

Therefore, the equality (2.3) implies

$$\sum_{i=1}^{m} \alpha_i \mu(P_i) = \sum_{i=1}^{m} \alpha_i \mu(P_i \cap H^+) = r. \tag{2.4}$$

P_1 has dimension n. Then there exist $2n$ hyperplanes $H_1, \ldots, H_{2n}$ such that

$$P_1 = \bigcap_{i=1}^{2n} H_i^+.$$

Replacing P_i with $P_i \cap H^+$ and iterating the above argument we get

$$\sum_{i=1}^{m} \alpha_i \mu(P_i \cap H_1^+ \cap H_2^+ \cap \cdots \cap H_{2^n}^+) = \sum_{i=1}^{m} \alpha_i \mu(P_i \cap P_1) = r. \tag{2.5}$$

and

$$\sum_{i=1}^{m} \alpha_i I_{P_i \cap P_1} = 0. \tag{2.6}$$

We repeat this argument with the remaining top dimensional boxes $P_2, \ldots, P_k$ and if we set

$$P_0 := P_1 \cap \cdots \cap P_k, \quad A := \alpha_1 + \cdots + \alpha_k$$

we conclude

$$\sum_{i=1}^{m} \alpha_i I_{P_i \cap P_0} = A I_{P_0} + \sum_{i>k} \alpha_i I_{P_i \cap P_0} = 0, \tag{2.7a}$$

$$\sum_{i=1}^{m} \alpha_i \mu(P_i \cap P_0) = A\mu(P_0) + \sum_{i>k} \alpha_i \mu(P_i \cap P_0) = r. \tag{2.7b}$$

In the above sums, at most one of the boxes is top dimensional, which contradicts the minimality of $k > 1$.

□

§2.3. **Continuous Invariant Valuations on** Pix**.**

Notation 2.6. We shall denote $\widetilde{T}_n$ the subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ generated by the translations and the permutations of coordinates in $\mathbb{R}^n$. □

Definition 2.7. A valuation μ on $\mathrm{Pix}(n)$ is called *invariant* if

$$\mu(gP) = \mu(P), \ \ \forall P \in \mathrm{Pix}(n), \ \ g \in \widetilde{T}_n.$$

and is called *translation invariant* if the same condition holds for all translations g. □

We aim to find all the invariant valuations on $\mathrm{Pix}(n)$. To avoid unnecessary complications, we will impose a further condition on the valuations.

The final condition we would like on our valuations on $\mathrm{Pix}(n)$ is that of continuity. Our valuations are functions on $\mathrm{Pix}(n)$, which is a collection of compact subsets of $\mathbb{R}^n$. So, in order for continuity to make any sense, we would like some concept of open sets for this collection of compact sets. A good way of achieving this goal is to make them into a metric space by defining a reasonable notion of distance.

Definition 2.8. (a) Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The *distance from* x *to* A, $d(x, A)$, is the nonnegative real number $d(x, A)$ defined by

$$d(x, A) = \inf_{a \in A} d(x, a)$$

where $d(x, a)$ is the Euclidian distance from x to a
(b) Let K and E be subsets of $\mathbb{R}^n$. Then the *Hausdorff distance* $\delta(K, E)$ is defined by:

$$\delta(K, E) = \max \left(\sup_{a \in K} d(a, E), \sup_{b \in E} d(K, b) \right).$$

(c) A sequence of compact sets K_n in $\mathbb{R}^n$ *converges* to a set K if $\delta(K_n, K) \longrightarrow 0$ as $n \longrightarrow \infty$. If this is the case, then we write $K_n \longrightarrow K$. □

Remark 2.9. If K and E are compact, then $\delta(K, E) = 0$ if and only if $K = E$. That is, the Hausdorff distance is positive definite on the set of compact sets in $\mathbb{R}^n$. □

Let $\mathbf{B}_n$ be the unit ball in $\mathbb{R}^n$. For $K \subset \mathbb{R}^n$ and $\epsilon > 0$, set

$$K + \epsilon \mathbf{B}_n := \{x + \epsilon u \mid x \in K \text{ and } u \in \mathbf{B}_n\}.$$

The following lemma (whose proof is clear) gives a hands-on understanding of how the Hausdorff distance behaves.

Lemma 2.10. *Let K and E be compact subsets of $\mathbb{R}^n$. Then*

$$\delta(K, E) \leq \epsilon \iff K \subset E + \epsilon \mathbf{B}_n \textit{ and } E \subset K + \epsilon \mathbf{B}_n. \quad □$$

The above result implies that the Hausdorff distance is actually a metric. The next result summarizes this.

Proposition 2.11. *The collection of compact subsets of $\mathbb{R}^n$ together with the Hausdorff distance forms a metric space.* □

Definition 2.12. Let $\mu : \mathrm{Pix}(n) \to \mathbb{R}$ be a valuation on $\mathrm{Pix}(n)$. Then μ is called (box) *continuous* if μ is continuous on boxes that is for any sequence of boxes P_i converging in the Hausdorff metric to a box P we have

$$\mu(P_i) \longrightarrow \mu(P) \qquad \square$$

We want to classify the continuous invariant valuations on $\mathrm{Pix}(n)$. We start by considering the problem in $\mathbb{R}^1$. An element of $\mathrm{Pix}(1)$ is a finite union of closed intervals. For $A \in \mathrm{Pix}(1)$, set

$$\mu_0^1(A) = \text{the number of connected components of } A$$

$$\mu_1^1(A) = \text{the length of } A$$

Both are continuous invariant valuations on $\mathrm{Pix}(1)$. It is clear that they are invariant under $\widetilde{T}_n$, which is, in this case, the group of translations. It is clear that both are continuous.

Proposition 2.13. *Every continuous invariant valuation $\mu : \mathrm{Pix}(1) \to \mathbb{R}$ is a linear combination of μ_0^1 and μ_1^1.*

Proof. Let $c = \mu(A)$, where A is a singleton set $\{x\}, x \in \mathbb{R}$. Now let $\mu' = \mu - c\mu_0^1$. μ' vanishes on points by construction. Now, define a continuous function $f : [0,\infty)\longrightarrow\mathbb{R}$ by $f(x) = \mu'([0,x])$. μ' is invariant because μ and μ_0^1 are invariant. Then, if A is a closed interval of length x, $\mu'(A) = f(x)$ since we can simply translate A to the origin.

Now observe that

$$f(x+y) = \mu'([0,x+y]) = \mu'([0,x] \cup \mu'([x,x+y]) = \mu'(0,x]) + \mu'([x,x+y]) - \mu'(\{x\})$$
$$= \mu'([0,x]) + \mu'([x,x+y]) = f(x) + f(y).$$

Since f is continuous and linear, we deduce that there exists a constant r such that $f(x) = rx$, for all $x \geq 0$. Therefore, $\mu' = r\mu_1^1$ and our assertion follows from the equality

$$\mu = \mu' + c\mu_0 = r\mu_1 + c\mu_0.$$

$\square$

We now move onto $\mathbb{R}^n$. Let $\mu_n(P)$ be the volume of a pixelation P of dimension n.

Definition 2.14. The *k-th elementary symmetric polynomial* in the variables $x_1, \ldots, x_n$ the polynomial $e_k(x_1, \ldots, x_n)$ such that $e_0 = 1$ and

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \quad 1 \leq k \leq n. \qquad \square$$

Observe that we have the identity

$$\prod_{j=1}^{n} (1 + tx_j) = \sum_{k=0}^{n} e_k(x_1, \cdots, x_n) t^k. \tag{2.8}$$

Theorem 2.15. *For $0 \leq k \leq n$, there exists a unique continuous valuation μ_k on $\mathrm{Pix}(n)$ invariant under $\widetilde{T}_n$, such that $\mu_k(P) = e_k(x_1, \ldots, x_n)$ whenever $P \in \mathrm{Par}(n)$ with sides of length $x_1, \ldots, x_n$.*

Proof. Let $\mu_0^1, \mu_1^1 : \mathrm{Pix}(1) \to \mathbb{R}$ be the valuations described above. We set $\mu_t^1 = \mu_0^1 + t\mu_1^1$, where t is a variable. Then,

$$\mu_t^n = \mu_t^1 \times \mu_t^1 \times \cdots \times \mu_t^1 : \mathrm{Par}(n) \to \mathbb{R}[t]$$

is an invariant valuation on parallelotopes with values in the ring of polynomials with real coefficients. By Groemer's Extension Theorem (2.5), μ_t^n extends to a valuation on $\mathrm{Pix}(n)$ which must also be invariant. Using (2.8) we deduce

$$\mu_t^n([0,x_1] \times \cdots \times [0,x_n]) = \prod_{j=1}^{n}(1+tx_j) = \sum_{k=0}^{n} e_k(x_1,\cdots,x_n)t^k.$$

For any parallelotope P we can write

$$\mu_t(P) = \sum_{k=0}^{n} \mu_k^n(P)t^k.$$

The coefficients $\mu_k^n(P)$ define continuous invariant valuations on $\mathrm{Par}(n)$ which extend to continuous invariant valuations on $\mathrm{Pix}(n)$ such that

$$\mu_t^n(S) = \sum_{k=0}^{n} \mu_k^n(S)t^k, \quad \forall S \in \mathrm{Pix}(n) \quad \mu_k^n([0,x_1] \times \cdots \times [0,x_n]) = e_k(x_1,\cdots,x_n).$$

□

Theorem 2.16. *The valuations μ_i on $\mathrm{Pix}(n)$ are normalized independently of the dimension n, i.e. $\mu_i^m(P) = \mu_i^n(P)$ for all $P \in \mathrm{Pix}(n)$.*

Proof. This follows from the preceding theorem and the definition of an elementary symmetric function. If $P \in \mathrm{Par}(n)$ and we consider the same $P \in \mathrm{Par}(k)$, where $k > n$, P remains a cartesian product of the same intervals, except there are some additional intervals of length 0, which do not effect μ_i. □

Since the valuation $\mu_k(P)$ is independent of the ambient space, μ_k is called the *k-th intrinsic volume*. μ_0 is the *Euler characteristic*. $\mu_0(Q)$=1 for all non-empty *boxes*.

Theorem 2.17. *Let H_1 and H_2 be complementary orthogonal subspaces of $\mathbb{R}^n$ spanned by subsets of the given coordinate system with dimensions h and $n-h$, respectively. Let P_i be a parallelotope in H_i and let $P = P_1 \times P_2$.*

$$\mu_i(P_1 \times P_2) = \sum_{r+s=i} \mu_r(P_1)\mu_s(P_2) \tag{2.9}$$

The identity is therefore valid when P_1 and P_2 are pixelations since both sides of the above equalities define valuations on $\mathrm{Par}(n)$ which extend uniquely to valuations on $\mathrm{Par}(n)$.

Proof. Suppose P_1 has sides of length $x_1,\ldots,x_h$ and P_2 has sides of length $y_1,\ldots,y_{n-h}$. Then,

$$\sum_{r+s=i} \mu_r(P_1)\mu_s(P_2) = \sum_{r+s=i}\left(\sum_{1\le j_1<\cdots<j_r\le h} x_{j_1}\cdots x_{j_h} \sum_{1\le k_1<\cdots<k_s\le n-h} y_{k_1}\cdots j_{k_s}\right)$$

Let $j_{r+1} = k_1 + h, \ldots, j_i = j_{r+s} = k_s + h$ and let $x_{h=1} = y_1, \ldots, x_n = y_{n-h}$, simply relabelling. Then,

$$\sum_{r+s=i} \mu_r(P_1)\mu_s(P_2) = \sum_{r+s=i} \left(\sum_{1 \le j_1 < \cdots < j_r \le h} x_{j_1} \cdots x_{j_r} \sum_{h+1 \le j_{r+1} < \cdots < j_i \le n} x_{j_{r+1}} \cdots x_{j_i} \right)$$
$$= \sum_{1 \le j_1 < \cdots < j_r < j_{r+1} < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i} = \mu_i(P_1 \times P_2)$$

since $P_1 \times P_2$ is simply a parallelotope of which we know how to compute μ_i. □

§2.4. **Classifying the Continuous Invariant Valuations on Pix.** At this point we are very close to a full description of the continuous invariant valuations on $\mathrm{Pix}(n)$.

Definition 2.18. A valuation μ on $\mathrm{Pix}(n)$ is said to be *simple* if $\mu(P) = 0$ for all P of dimension less than n. □

Theorem 2.19 (Volume Theorem for $\mathrm{Pix}(n)$)**.** *Let μ be a translation invariant, simple valuation defined on $\mathrm{Par}(n)$ and suppose that μ is either continuous or monotone. There exists $c \in \mathbb{R}$ such that $\mu(P) = c\mu_n(P)$ for all $P \in \mathrm{Pix}(n)$, that is μ is equal to the volume, up to a constant factor.*

Proof. Let $[0,1]^n$ denote the unit cube in $\mathbb{R}^n$ and let $c = \mu([0,1])^n$. Then, $\mu\left(\left[0, \frac{1}{k}\right]^n\right) = \frac{c}{k^n}$ for all $k > 0 \in \mathbb{Z}$. Therefore, $\mu(C) = c\mu_n(C)$ for every box C of rational dimensions with sides parallel to the coordinate axes since C can be built from $[0, \frac{1}{k}]^n$ cubes for some k. Since μ is either continuous or monotone and $\mathbb{Q}$ is dense in $\mathbb{R}$, then $\mu(C) = c\mu_n(C)$ for C with real dimensions since we can find a sequence of rational C_n converging to C. Then, by inclusion-exclusion, $\mu(P) = c\mu_n(P)$ for all $P \in \mathrm{Pix}(n)$ (since it works for parallelotopes, we can extend it to a valuation on pixelations). □

Theorem 2.20. *The valuations $\mu_0, \mu_1, \ldots, \mu_n$ form a basis for the vector space of all continuous invariant valuations defined on $\mathrm{Pix}(n)$.*

Proof. Let μ be a continuous invariant valuation on $\mathrm{Pix}(n)$. Denote by $x_1, \ldots, x_n$ be the standard Euclidean coordinates on $\mathbb{R}^n$ and let H_j denote the hyperplane defined by the equation $x_j = 0$. The restriction on μ to H_j is an invariant valuation on pixelations in H_j. Proceeding by induction (taking $n = 1$ as a base case, which was proven in Proposition 2.13, assume

$$\mu(A) = \sum_{i=0}^{n-1} c_i \mu_i(A) \quad \forall A \in \mathrm{Pix}(n) \text{ such that } A \subseteq H_j \tag{2.10}$$

The c_i are the same for all H_j since $\mu_0, \ldots, \mu_{n-1}$ are invariant under permutation. Then, $\mu - \sum_{i=0}^{n-1} c_i\mu_i$ vanishes on all lower dimensional pixelations in $\mathrm{Pix}(n)$ since any such pixelation is in a hyperplane parallel to one of the H_j's (since the μ_i's are translationally invariant). Then, by Theorem 2.19 we deduce

$$\mu - \sum_{i=0}^{n-1} c_i\mu_i = c_n\mu_n$$

which proves our claim. □

If we can find a continuous invariant valuation on $\mathrm{Pix}(n)$, then we know that it is a linear combination of the μ_i's. However, we would like a better description, if at all possible. The following corollary yields one.

Definition 2.21. A valuation μ is said to be *homogenous* of degree $k > 0$ if $\mu(\alpha P) = \alpha^k \mu(P)$ for all $P \in \mathrm{Pix}(n)$ and all $\alpha \geq 0$.

Corollary 2.22. *Let μ be a continuous invariant valuation defined on $\mathrm{Pix}(n)$ that is homogenous of degree k for some $0 \leq k \leq n$. Then there exists $c \in \mathbb{R}$ such that $\mu(P) = c\mu_k(P)$ for all $P \in \mathrm{Pix}(n)$.*

Proof. There exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $\mu = \sum_{i=0}^{n} c_i \mu_i$. If $P = [0,1]^n$, then for all $\alpha > 0$,

$$\mu(\alpha P) = \sum_{i=0}^{n} c_i \mu_i(\alpha P) = \sum_{i=0}^{n} c_i \alpha^i \mu_i(P) = \sum_{i=0}^{n} \binom{n}{i} c_i \alpha_i$$

Meanwhile,

$$\mu(\alpha P) = \alpha^k \mu(P) = \alpha^k \sum_{i=0}^{n} c_i \mu_i(P) = \alpha^k \sum_{i=0}^{n} c_i \binom{n}{i}$$

so

$$\left(\sum_{i=0}^{n} c_i \binom{n}{i} \right) \alpha^k = \sum_{i=0}^{n} c_i \binom{n}{i} \alpha^i$$

meaning that $c_i = 0$ for $i \neq k$. □

3. **Valuations on polyconvex sets**

Now that we understand continuous invariant valuations on a very specific collection of subsets of $\mathbb{R}^n$, we recognize the limitations of this viewpoint. Notably we have said nothing of valuations on exciting shapes such as triangles and disks. To include these, we dramatically expand our collection of subsets and again try to classify the continuous invariant valuations. This effort will turn out to be a far greater undertaking.

§3.1. **Convex and Polyconvex Sets.**

Definition 3.1. (a) $K \subset \mathbb{R}^n$ is *convex* if for any two points x and y in K, the line segment between x and y lies in K. We denote by $\mathcal{K}^n$ the set of all compact convex subsets of $\mathbb{R}^n$.
(b) A *polyconvex* set is a finite union of compact convex sets. We denote by $\mathrm{Polycon}(n)$ the set of all polyconvex sets in $\mathbb{R}^n$. □

Example 3.2. A single point is a compact convex set. If

$$T := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1\}.$$

then T is a filled in triangle in $\mathbb{R}^2$, so that $T \in \mathcal{K}^2$. Also, ∂T is a polyconvex set, but not a convex set. □

One of the most important properties of convex sets is the *separation property*.

Proposition 3.3. *Suppose $C \subset \mathbb{R}^n$ is a closed convex set and $x \in \mathbb{R}^n \setminus C$. Then there exists a hyperplane which separates x from C, i.e. x and C lie in different half-spaces determined by the hyperplane.*

Proof. We outline only the main geometric ideas of the construction of such a hyperplane. Let

$$d := \mathrm{dist}(x, C).$$

Then we can find a *unique* point $y \in C$ such that $d = |y - x|$. The hyperplane perpendicular to the segment $[x, y]$ and intersecting it in the middle will do the trick. □

Definition 3.4. If A is a polyconvex set in $\mathbb{R}^n$, then we say that A is of *dimension n* or has *full dimension* if A is not contained in a finite union of hyperplanes. Otherwise, we say that A has *lower dimension*. □

Remark 3.5. $\mathrm{Polycon}(n)$ is a distributive lattice under union and intersection. Furthermore, $\mathcal{K}^n$ is a generating set of $\mathrm{Polycon}(n)$. □

We must now explore some tools we can use to understand these compact, convex sets. The most critical of these is the *support function*. Let $\langle -, - \rangle$ denote the standard inner product on $\mathbb{R}^n$ and by $| - |$ the associated norm. We set

$$\mathbf{S}^{n-1} := \{u \in \mathbb{R}^n; \ \ |u| = 1\}.$$

Definition 3.6. Let $K \in \mathcal{K}^n$ and nonempty. Then its *support function*, $h_K : \mathbf{S}^{n-1} \to \mathbb{R}$, given by

$$h_K(u) := \max_{x \in K} \langle u, x \rangle. \qquad \square$$

Example 3.7. If $K = \{x\}$ is just a single point, then $h_K(u) = \langle u, x \rangle$ for all $u \in \mathbf{S}^{n-1}$.

Remark 3.8. (a) We can characterize the support function in terms of a function on $\mathbb{R}^n$ as follows. Let $\tilde{h} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be such that $\tilde{h}(tu) = t\tilde{h}(u)$ for all $u \in \mathbf{S}^{n-1}$ and $t \geq 0$. Let h be the restriction of $\tilde{h}$ to $\mathbf{S}^{n-1}$. Then h is a support function of a compact convex set in $\mathbb{R}^n$ if and only if

$$\tilde{h}(x + y) \leq \tilde{h}(x) + \tilde{h}(y)$$

for all $x, y \in \mathbb{R}^n$.
(b) Consider the hyperplane $H(K, u) = \{x \in \mathbb{R}^n \mid \langle x, u \rangle = h_K(u)\}$ and the closed half-space $H(K, u)^- = \{x \in \mathbb{R}^n \mid \langle x, u \rangle \leq h_K(u)\}$. Then it is easy to see that $H(K, u)$ is "tangent" to ∂K and that K lies wholly in $H(K, u)^-$ for all $u \in \mathbf{S}^{n-1}$. The separation property described in Proposition 3.3 implies

$$K = \bigcap_{u \in \mathbf{S}^{n-1}} H(K, u)^-.$$

In other words, K is *uniquely* determined by its support function. $\square$

Definition 3.9. Let K, L be in $\mathcal{K}^n$. Then we define

$$K + L := \{x + y \mid x \in K, y \in L\}$$

and call $K + L$ the *Minkowski sum* of K and L.

Remark 3.10. We want to point out that for every $K, L \in \mathcal{K}^n$ we have

$$K \subset L \Longleftrightarrow h_k(u) \leq h_k(u), \quad \forall u \in \mathbf{S}^{n-1}$$

and

$$\begin{aligned} h_{K+L}(u) &= \max_{x \in K, y \in L} (\langle x + y, u \rangle) = \max_{x \in K, y \in L} (\langle x, u \rangle + \langle y, u \rangle) \\ &= \max_{x \in K} (\langle x, u \rangle) + \max_{y \in L} (\langle y, u \rangle) = h_K(u) + h_L(u). \end{aligned} \qquad \square$$

Remark 3.11. Recall that for compact sets K and L in $\mathbb{R}^n$, the Hausdorff metric satisfies $\delta(K, L) \leq \epsilon$ if and only if $K \subset L + \epsilon B$ and $L \subset K + \epsilon B$. In light of this fact and the preceding comments, one can show that

$$\delta(K, L) = \sup_{u \in \mathbf{S}^{n-1}} |h_K(u) - h_L(u)|.$$

That is, the Hausdorff metric on compact convect subsets of $\mathbb{R}^n$ is given by the uniform metric on the set of support functions of compact convex sets. $\square$

Now that we have some tools to understand these compact convex sets, we will soon wish to consider valuations on them. But we don't want just any valuations. We would like our valuations to be somehow tied to the shape of the convex set alone, rather than where the convex set is in the space. So, we would like some sort of invariance under types of transformations. Furthermore, to make things nicer we would like to restrict our attention to those valuations which are somehow continuous. We formalize these notions.

It is easiest to begin with continuity. Recall that the Hausdorff distance turns the set of compact subsets of $\mathbb{R}^n$ into a metric space. Since $\mathcal{K}^n$ is a subset of these, continuity is a well defined concept for elements of $\mathcal{K}^n$. (We restrict our attention to these elements and not all of $\mathrm{Polycon}(n)$ since dimensionality problems arise when considering limits of polyconvex sets.)

Definition 3.12. A valuation $\mu : \mathrm{Polycon}(n) \to \mathbb{R}$ is said to be *convex continuous* (or simply continuous where no confusion is possible) if

$$\mu(A_n) \longrightarrow \mu(A)$$

whenever A_n, A are compact, convex sets and $A_n \longrightarrow A$. □

Notation 3.13. Let E_n be the Euclidean group of $\mathbb{R}^n$, which is the subgroup of affine transformations of $\mathbb{R}^n$ generated by translations and rotations. For any $g \in E_n$ there exist $T \in SO(n)$ and $v \in \mathbb{R}^n$ such that

$$g(x) = T(x) + v, ; \ \forall x \in \mathbb{R}^n.$$

The elements of E_n are also known as *rigid motions*.

Definition 3.14. Let $\mu : \mathrm{Polycon}(n) \to \mathbb{R}$ be a valuation. Then μ is said to be *rigid motion invariant* (or invariant when no confusion is possible) if

$$\mu(A) = \mu(gA)$$

for all $A \in Polycon(n)$ and $g \in E_n$. If the same holds only for translations g then μ is said to be *translation invariant*. □

Our aim is to understand the set of convex-continuous invariant valuations on $\mathrm{Polycon}(n)$, which we will denote by $\mathbf{Val}(n)$. Note that for every $m \leq n$ we have an inclusion

$$\mathrm{Polycon}(m) \subset \mathrm{Polycon}(n)$$

given by the natural inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$. In particular, any continuous invariant valuation $\mu : \mathrm{Polycon}(n) \to \mathbb{R}$ induces by restriction a valuation on $\mathrm{Polycon}(m)$. In this way we obtain for every $m \leq n$ a restriction map

$$S_{m,n} : \mathbf{Val}(n) \to \mathbf{Val}(m)$$

such that $S_{n,n}$ is the identity map and for every $k \leq m \leq n$ we have $S_{k,n} = S_{k,m} \circ S_{m,n}$, i.e. the diagram below commutes.

$$\begin{array}{ccc} \mathbf{Val}(n) & \xrightarrow{S_{m,n}} & \mathbf{Val}(m) \\ & {}_{S_{k,n}}\searrow & \downarrow S_{k,m} \\ & & \mathbf{Val}(k) \end{array}$$

Definition 3.15. An *intrinsic valuation* is a *sequence of convex-continuous, invariant* valuations $\mu^n \in \mathbf{Val}(n)$ such that for every $m \leq n$ we have

$$\mu^m = S_{m,n}\mu^n.$$

□

Remark 3.16. (a) To put the above definition in some perspective we need to recall a classical notion. A *projective sequence* of Abelian groups is a sequence of Abelian groups $(G_n)_{n\geq 1}$ together with a family of group morphisms

$$S_{m,n} : G_n \to G_m, \quad m \leq n$$

satisfying

$$S_{nn} = \mathbb{1}_{G_n}, \;\; S_{kn} = S_{km} \circ S_{mn}, \;\; \forall k \leq m \leq n..$$

The *projective limit* of a projective sequence $\{G_n; \;\; S_{mn}\}$ is the subgroup

$$\operatorname{limproj}_n G_n \subset \prod_n G_n$$

consisting of sequences $(g_n)_{n\geq 1}$ satisfying

$$g_n \in G_n, \;\; g_m = S_{m,n}g_n, \;\; \forall m \leq n.$$

The sequence $(\mathbf{Val}(n))$ together with the maps S_{mn} define a projective sequence and we set

$$\mathbf{Val}(\infty) := \operatorname{limproj}_n \mathbf{Val}(n) \subset \prod_{n\geq 0} \mathbf{Val}(n).$$

An intrinsic measure is then an element of $\mathbf{Val}(\infty)$.

Similarly if we denote by $\mathbf{Val}_{\mathrm{Pix}}(n)$ the space of continuous, invariant valuations on $\mathrm{Pix}(n)$ the we obtain again a projective sequence of vector spaces and an element in the corresponding projective limit will be an intrinsic valuation in the sense defined in the previous section.

(b) Observe that since $\mathbf{Val}_{\mathrm{Pix}}(n) \subset \mathbf{Val}(n)$ we have a natural map

$$\Phi_n : \mathbf{Val}(n) \to \mathbf{Val}_{\mathrm{Pix}}(n).$$

A priori this linear map need be neither injective nor surjective. However, in a later section we will show that this map is a linear isomorphism. □

§3.2. **Groemer's Extension Theorem.** We now show that any convex-continuous valuation on $\mathcal{K}^n$ can be extended to $\mathrm{Polycon}(n)$. Thus, we can confine our studies to continuous valuations on compact, convex sets.

Theorem 3.17. *A convex, continuous valuation μ, on $\mathcal{K}^n$ can be extended (uniquely) to* $\mathrm{Polycon}(n)$. *Moreover, if μ is also invariant, then so is its extension.*

Proof. Suppose that μ is a convex-continuous valuation on $\mathcal{K}^n$. In light of Groemer's integral theorem, we need only show that the integral defined by μ on the space of indicator functions is well defined.

We proceed by induction on dimension. In dimension zero, this proposition is trivial, and in dimension one, $\mathcal{K}^n$ is the same as $\mathrm{Par}(n)$ and $\mathrm{Polycon}(n)$ is the same as $\mathrm{Pix}(n)$. Hence, we have already done this dimension as well. So, suppose the theorem holds for dimension $n-1$.

Suppose for the sake of contradiction that the integral defined by μ is not well defined. Using the same technique as in Theorem 2.5, Groemer's Extension Theorem for parallelotopes, suppose that there exist $K_1, \ldots, K_m \in \mathcal{K}^n$ such that

$$\sum_{i=1}^{m} \alpha_i I_{K_i} = 0 \tag{3.1}$$

while

$$\sum_{i=1}^{m} \alpha_i \mu(K_i) = r \neq 0 \tag{3.2}$$

Take m to be the least positive integer such that (3.1) and (3.2) exist.

Choose a hyperplane H with associated closed half-spaces H^+ and H^- such that $K_1 \subset \mathrm{Int}(H^+)$. Recall that $I_{A\cap B} = I_A I_B$. Thus, in light of equation (3.1), we can multiply and get

$$\sum_{i=1}^{m} \alpha_i I_{K_i \cap H^+} = 0$$

as well as

$$\sum_{i=1}^{m} \alpha_i I_{K_i \cap H^-} = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i I_{K_i \cap H} = 0.$$

Now note that $K_i = (K_i \cap H^+) \cup (K_i \cap H^-)$ and that $H^+ \cap H^- = H$. Thus, since μ is a valuation, we may apply this decomposition and see that

$$\sum_{i=1}^{m} \alpha_i \mu(K_i) = \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^+) + \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^-) - \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H).$$

Since each $K_i \cap H$ lies inside H, a space of dimension $n-1$, we deduce from the induction assumption that

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap H) = 0.$$

Moreover, since we took $K_1 \subset \mathrm{Int}\,(H^+)$, we have

$$I_{K_1 \cap H^-} = 0,$$

and the sum

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^-) = \sum_{i=2}^{m} \alpha_i \mu(K_i \cap H^-)$$

must be zero due to the minimality of m. Thus, from (3.2), we have

$$0 \neq r = \sum_{i=1}^{m} \alpha_i \mu(K_i) = \sum_{i=1}^{m} \alpha_i \mu(K_i \cap H^+).$$

Thus we have replaced K_i by $K_i \cap H^+$ in (3.1) and (3.2). We now repeat this process. By taking a countable dense subset, choose a sequence of hyperplanes, $H_1, H_2, \ldots$ such that $K_1 \subset \mathrm{Int}(H_i^+)$, and

$$K_1 = \bigcap H_i^+.$$

Thus, iterating the proceeding argument, we have that

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap H_1^+ \cap \cdots \cap H_q^+) = r \neq 0$$

for all $q \geq 1$. Since μ is continuous, we can take the limit as $q \to \infty$, giving

$$\sum_{i=1}^{m} \alpha_i \mu(K_i \cap K_1) = r \neq 0,$$

while applying the same type of argument to (3.1) yields

$$\sum_{i=1}^{m} \alpha_i I_{K_i \cap K_1} = 0.$$

Thus, we find ourselves in exactly the same position we were with equations (3.1) and (3.2); therefore, we may repeat the entire argument for $K_2, \ldots, K_m$, giving

$$\sum_{i=1}^{m} \alpha_i \mu(Ki \cap K_1 \cap \cdots \cap K_m) = \sum_{i=1}^{m} \alpha_i \mu(K_1 \cap \cdots \cap K_m)$$

$$= \left(\sum_{i=1}^{m} \alpha_i \right) \cdot \mu(K_1 \cap \cdots \cap K_m) = r \neq 0 \tag{3.3a}$$

and

$$\sum_{i=1}^{m} \alpha_i I_{K_1 \cap \cdots \cap K_m} = \left(\sum_{i=1}^{m} \alpha_i \right) (I_{K_1 \cap \cdots \cap K_m}) = 0. \tag{3.4}$$

The equalities (3.3a) and (3.4) contradict each other. The first implies that $\alpha_1 + \cdots + \alpha_m \neq 0$ and $K_1 \cap \cdots \cap K_m \neq \emptyset$, while from the second implies that $\alpha_1 + \cdots + \alpha_m = 0$ or $I_{K_1 \cdots \cap K_m} = 0$.

Thus, the integral must be well defined, so there exists a unique extension of μ to Polycon(n). □

§3.3. **The Euler Characteristic.**

Theorem 3.18. *(a) There exists an* intrinsic valuation $\mu_0 = (\mu_0^n)_{n \geq 0}$ *uniquely determined by*

$$\mu_0^n(C) = 1, \quad \forall C \in \mathcal{K}^n.$$

(b) Suppose $C \in \mathrm{Polycon}(n)$*, and* $u \in \mathbf{S}^{n-1}$*. Define* $\ell_u : \mathbb{R}^n \to \mathbb{R}$ *by* $\ell_u(x) = \langle u, x \rangle$ *and set*

$$H_t := \ell_u^{-1}(t), \quad C_t := C \cap H_t, \quad F_{u,C}(t) := \mu_0(C_t).$$

Then $F_{u,C}(t)$ *is a* simple function, *i.e. it is a linear combination of integral coefficients of characteristic functions* I_S, $S \in \mathcal{K}^1$. *Moreover*

$$\mu_0(C) = \int I_C d\mu_0 = \int F_{u,C}(t) d\mu_0(t) = \sum_t \big(F_{u,C}(t) - F_{u,C}(t+0) \big). \qquad \textbf{(Fubini)}$$

Proof. Part (a) follows immediately from Theorem 3.17. To establish the second part we use again Theorem 3.17. For every $C \in \mathrm{Polycon}(n)$ define

$$\lambda_u(C) := \int F_{u,C}(t) d\mu_0(t)$$

Observe that for every $t \in \mathbb{R}$ and every $C_1, C_2 \in \mathrm{Polycon}(n)$ we have

$$F_{u,C_1 \cup C_2}(t) = \mu_0\big(H_t \cap (C_1 \cup C_2) \big) = \mu_0\big((H_t \cap C_1) \cup (H_t \cap C_2) \big)$$
$$= \mu_0(H_t \cap C_1) + \mu_0(H_t \cap C_2) - \mu_0\big((H_t \cap C_1) \cap (H_t \cap C_2) \big)$$
$$= F_{u,C_1}(t) + F_{u,C_2}(t) - F_{u,C_1 \cap C_2}(t).$$

Observe that if $C \in \mathcal{K}^n$ then $F_{u,C}$ is the characteristic function of a compact interval $\subset \mathbb{R}^1$. This shows that $\chi_{u,C}$ is a simple function for every $C \in \mathrm{Polycon}(n)$. Moreover

$$\int F_{u,C_1 \cup C_2} d\mu_0 = \int F_{u,C_1} d\mu_0 + \int F_{u,C_2} d\mu_0 - \int F_{u,C_1 \cap C_2} d\mu_0$$

so that the correspondence

$$\mathrm{Polycon}(n) \ni C \mapsto \lambda_u(C)$$

is a valuation such that $\lambda_u(C) = 1$, $\forall C \in \mathcal{K}^n$. From part (a) we deduce

$$\int F_{u,C} d\mu_0 = \lambda_u(C) = \mu_0(C).$$

We only have to prove that for every simple function $h(t)$ on $\mathbb{R}^1$ we have

$$\int h(t) d\mu_0(t) = L_1(h) := \sum_t \big(h(t) - h(t+0) \big).$$

To achieve this observe that $L_1(h)$ is linear and convex continuous in h and thus defines an integral on the space of simple functions. Moreover if h is the characteristic function of a compact interval then $L_1(h)$. Thus by Theorem 3.17 we deduce

$$L_1 = \int d\mu_0.$$

□

Remark 3.19. Denote by $\mathcal{S}(\mathbb{R}^n)$ the the Abelian subgroup of $\mathrm{Map}(\mathbb{R}^n, \mathbb{Z})$ generated by indicator functions of compact convex subsets of $\mathbb{R}^n$. We will refer to the elements of $\mathcal{S}(\mathbb{R}^n)$ as *simple functions*. Thus any $f \in \mathcal{S}(\mathbb{R}^n)$ can be written non-uniquely as a sum

$$f = \sum_i \alpha_i I_{C_i}, \quad \alpha_i \in \mathbb{Z}, \quad C_i \in \mathcal{K}^n.$$

The Euler characteristic then defines an integral

$$\int d\mu_0 : \mathcal{S}(\mathbb{R}^n) \to \mathbb{Z}, \quad \int \Big(\sum_i \alpha_i I_{C_i} \Big) d\mu_0 = \sum_i \alpha_i.$$

The *convolution* of two simple functions f, g is the function $f * g$ defined by

$$f * g(x) := \int \bar{f}_x(y) \cdot g(y) d\mu_0(y),$$

where

$$\bar{f}_x(y) := f(x-y).$$

Observe that if $A, B \in \mathcal{K}^n$ and $A+B$ is their Minkowski sum then

$$I_A * I_B = I_{A+B}$$

so that $f * g$ is a simple function for any $f, g \in \mathcal{S}(\mathbb{R}^n)$. Moreover,

$$f * g = g * f, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

Finally, $f * I_{\{0\}} = f$, for all $f \in \mathcal{S}(\mathbb{R}^n)$ so that $\big(\mathcal{S}(\mathbb{R}^n), +, * \big)$ is a commutative ring with 1.□

Definition 3.20. A *convex polyhedron* is an intersection of a finite collection of closed half-spaces. A *convex polytope* is a *compact convex* polyhedron. A *polytope* is a finite union of convex polytopes. The polytopes form a distributive sublattice of $\mathrm{Polycon}(n)$.

The dimension of a convex polyhedron P is the dimension of the affine subspace $\mathrm{Aff}(P)$ generated by P. We denote by $\mathrm{relint}(P)$ the relative interior of P that is, the interior of P relative to the topology of $\mathrm{Aff}(P)$.

Remark 3.21. Give a convex polytope P, the boundary ∂P is also a polytope. Therefore, $\mu_0(\partial P)$ is defined.

Theorem 3.22. *If $P \subset \mathbb{R}^n$ is a convex polytope of dimension $n > 0$, then $\mu_0(\partial P) = 1 - (-1)^n$.*

Proof. Let $u \in \mathbf{S}^{n-1}$ be a unit vector and define $\ell_u : \mathbb{R}^n \to \mathbb{R}$ as before. Using the previous notation, note that $H_t \cap \partial P = \partial(H_t \cap P)$ if t is not a boundary point of the interval $\ell_u(P) \subset \mathbb{R}$.

Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(t) = \mu_0(H_t \cap \partial P).$$

We proceed by induction. For $n = 1$, we have $\mu_0(\partial P) = 2 = 1 - (-1)$ since ∂P consists of two distinct points (since P is an interval).

For $n > 1$, it follows from the induction by hypothesis that

$$\mu_0(H_t \cap \partial P) = \mu_0(\partial(H_t \cap P)) = 1 - (-1)^{n-1} \tag{*}$$

if $t \in \ell_u(P)$ is not a boundary point of the interval $\ell_u(P)$. If $t \in \partial \ell_u(P)$, we have

$$\mu_0(H_t \cap \partial P) = 1 \tag{**}$$

since $H_t \cap P$ is a face of P and is thus in $\mathcal{K}^n$. Finally,

$$\mu_0(H_t \cap \partial P) = 0 \tag{***}$$

when $H_t \cap \partial P = \emptyset$.

We can now compute

$$\int F(t)d\mu_0(t) = \sum_t (F(t) - F(t+0))$$

which vanishes except at the two point a and b ($a < b$) where $[a, b] = \ell_u(P)$. Then,

$$\sum_t (F(t) - F(t+0)) = F(a) - F(a+0) + F(b) - F(b+0).$$

Now observe that $F(b+0) = 0$ by (***), $F(b) = F(a) = 1$ by (**) and $F(a+0) = 1 - (-1)^{n+1}$ by (*). Then,

$$\int F(t)d\mu_0(t) = 1 - 1 + (-1)^{n-1} + 1 = (1) + (-1)^{n-1} = 1 - (-1)^n$$

□

Theorem 3.23. *Let P be a compact convex polytope of dimension k in $\mathbb{R}^n$. Then*

$$\mu_0(\mathrm{relint}(P)) = (-1)^k \tag{3.5}$$

Proof. Since μ_0 is normalized independently of n, we can consider P in the k-dimensional plane in $\mathbb{R}^n$ in which it is contained. Then, $\mathrm{relint}(P) = P \setminus \partial P$ so

$$\mu_0(\mathrm{relint}(P)) = \mu_0(P) - \mu_0(\partial P) = (-1)^k.$$

□

Definition 3.24. A *system of faces* of a polytope P, is a family $\mathcal{F}$ of convex polytopes such that the following hold.

(a) $\bigcup_{Q\in\mathcal{F}} \mathrm{relint}(Q) = P$.

(b) If $Q, Q' \in \mathcal{F}$ and $Q \neq Q'$, then $\mathrm{relint}(Q) \cap \mathrm{relint}(Q') = \emptyset$. □

Theorem 3.25 (Euler-Schläfli-Poincaré)**.** *Let $\mathcal{F}$ be a system of faces of the polytope P, and let f_i be the number of elements in $\mathcal{F}$ of dimension i. Then, $\mu_0 = f_0 - f_1 + f_2 - \cdots$.*

Proof. We have

$$I_P = \sum_{Q\in\mathcal{F}} I_{\mathrm{relint}(Q)}$$

so that

$$\mu_0(P) = \sum_{Q\in\mathcal{F}} \mu_0(\mathrm{relint}(Q)) = \sum_{Q\in\mathcal{F}} (-1)^{\dim Q} = \sum_{k\geq 0} (-1)^k f_k.$$

□

§3.4. **Linear Grassmannians.** We denote by $\mathrm{Gr}(n,k)$ the set of all k-dimensional subspaces in $\mathbb{R}^n$. Observe that the orthogonal group $O(n)$ acts transitively on $\mathrm{Gr}(n,k)$ and the stabilizer of a k-dimensional coordinate plane can be identified with the Cartesian product $O(k)\times O(n-k)$. Thus $\mathrm{Gr}(n,k)$ can also be identified with the space of left cosets $O(n)/\big(O(k)\times O(n-k)\big)$.

Example 3.26. $\mathrm{Gr}(n,1)$ is the set of lines in $\mathbb{R}^n$, a.k.a the projective space $\mathbb{RP}^{n-1}$. Denote by $\mathbf{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We have a natural map

$$\ell:\mathbf{S}^{n-1}\to\mathrm{Gr}(n,1),\;\; \mathbf{S}^{n-1}\ni x\mapsto \ell(x)=\text{the line through } 0 \text{ and } x.$$

The map ℓ is 2-to-1 since

$$\ell(x)=\ell(-x).$$

□

$\mathrm{Gr}(n,k)$ can also be viewed as a subset of the vector space of symmetric linear operators $\mathbb{R}^n\to\mathbb{R}^n$ via the map

$$\mathrm{Gr}(n,k)\ni V\mapsto P_V=\text{the orthogonal projection onto } V.$$

As such it is closed and bounded and thus compact. To proceed further we need some notations and a classical fact.

Denote by ω_n the volume of the unit ball in $\mathbb{R}^n$ and by σ_{n-1} the $(n-1)$-dimensional "surface area" of $\mathbf{S}^{n-1}$. Then

$$\sigma_{n-1}=n\omega_{n-1}$$

and

$$\omega_n=\frac{\pi^{n/2}}{\Gamma(n/2+1)}=\begin{cases}\dfrac{\pi^k}{k!} & n=2k\\[2ex] \dfrac{2^{2k+1}\pi^k k!}{(2k+1)!} & n=2k+1\end{cases},$$

where $\Gamma(x)$ is the gamma function. We list below the values of ω_n for small n.

n	0	1	2	3	4
ω_n	1	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$

Theorem 3.27. *For every positive constant c there exists a* unique *measure $\mu=\mu_c$ on $\mathrm{Gr}(n,k)$ which is invariant, i.e.*

$$\mu(g\cdot S)=\mu(S),\;\;\forall g\in O(n),\;\; S\subset\mathrm{Gr}(n,k)\;\; \textit{open subset}$$

and has finite volume, i.e. $\mu(\mathrm{Gr}(n,k))=c$. □

For a proof we refer to [**San**].

Example 3.28. Here is how we construct such a measure on $\mathrm{Gr}(n,1)$. Given an open set $U\subset\mathrm{Gr}(n,1)$ we obtain a subset

$$\widetilde{U}=\ell^{-1}(U)\subset\mathbf{S}^{n-1}$$

$\widetilde{U}$ consists of two diametrically opposed subsets of the unit sphere. Define

$$\mu(U) = \frac{1}{2}\,\text{area}(\widetilde{U})$$

Observe that for this measure

$$\mu(\text{Gr}(n,1)) = \frac{1}{2}\,\text{area}(\mathbf{S}^{n-1}) = \frac{\sigma_{n-1}}{2} = \frac{n\omega_n}{2}.$$

□

Observe that a constant multiple of an invariant measure is an invariant measure. In particular

$$\mu_c = c \cdot \mu_1.$$

Define

$$[n] := \frac{n\omega_n}{2\omega_{n-1}}, \quad [n]! := [1]\cdot[2]\cdots[n] = \frac{n!}{2^n}\omega_n, \quad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]}{[k]![n-k]!}$$

and denote by ν_k^n the invariant measure on $\text{Gr}(n,k)$ such that

$$\nu_k^n(\text{Gr}(n,k)) = \begin{bmatrix} n \\ k \end{bmatrix}.$$

§3.5. **Affine Grassmannians.** We denote by $\text{Graff}(n,k)$ the space of k-dimensional affine subspaces (planes) of $\mathbb{R}^n$. For every affine k-plane we denote by $\Pi(V)$ the linear subspace parallel to V, and by $V^\perp$ the orthogonal complement of $\Pi(V)$. We obtain in this fashion a surjection

$$\Pi : \text{Graff}(n,k) \to \text{Gr}(n,k), \quad V \mapsto \Pi(V)$$

Observe that an affine k-plane V is uniquely determined by $V^\perp$ and the point $p = V^\perp \cap V$. The fiber of the map $\Pi : \text{Graff}(n,k) \to \text{Gr}(n,k)$ over a point $L \in \text{Gr}(n,k)$ is the set $\Pi^{-1}(L)$ consisting of all affine k-planes parallel to L. This set can be canonically identified with $L^\perp$ via the map

$$\Pi^{-1}(L) \ni V \mapsto V \cap L^\perp \in L^\perp.$$

The map $\Pi : \text{Graff}(n,k) \to \text{Gr}(n,k)$ is an example of vector bundle.

We now describe how to integrate functions $f : \text{Graff}(n,k) \to \mathbb{R}$. Define

$$\int_{\text{Graff}(n,k)} f(V)\,d\lambda_k^n := \int_{\text{Gr}(n,k)} \Big(\int_{L^\perp} f(L+p)\,d_{L^\perp}p \Big) d\nu_k^n,$$

where $d_{L^\perp}p$ denotes the Euclidean volume element on $L^\perp$.

Example 3.29. Let

$$f : \text{Gr}(3,1) \to \mathbb{R}, \quad f(L) := \begin{cases} 1 & \text{dist}\,(L,0) \le 1 \\ 0 & \text{dist}\,(L,0) > 1 \end{cases}.$$

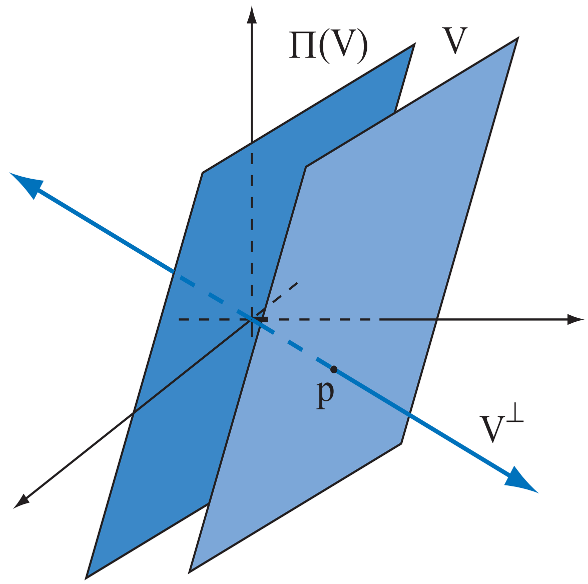

FIGURE 2. *The orthogonal projection of an element in* $\mathrm{Graff}(3,2)$.

Observe that f is none other than the indicator function of the set $\mathrm{Graff}(3,1;\mathbb{B}^3)$ of affine lines in $\mathbb{R}^3$ which intersect $\mathbb{B}^3$, the closed unit ball centered at the origin. Then

$$\int_{\mathrm{Graff}(3,1)} f(V)d\lambda_1^3 = \int_{\mathrm{Gr}(3,1)} \underbrace{\left(\int_{L^\perp} f(L+p)dp\right)}_{=\omega_2} d\nu_1^3 = \omega_2 \begin{bmatrix}3\\1\end{bmatrix} = [3]\cdot\omega_2$$

$$= \frac{3\omega_3}{2\omega_2}\omega_2 = \frac{3\omega_3}{2} = 2\pi.$$

In particular

$$\lambda_1^3\big(\,\mathrm{Graff}(3,1;\mathbb{B}^3)\,\big) = 2\pi = \frac{\sigma_2}{2} = \frac{1}{2} \times \text{surface area of } \mathbb{B}^3.$$

□

§3.6. **The Radon Transform.** At this point, we further our goal of understanding $\mathbf{Val}(n)$ by constructing some of its elements.

Recall that $\mathcal{K}^n$ is the collection of compact convex subsets of $\mathbb{R}^n$ and $\mathrm{Polycon}(n)$ is the collection of polyconvex subsets of $\mathbb{R}^n$. We denote by $\mathbf{Val}(n)$ the vector space of convex continuous, rigid motion invariant valuations

$$\mu : \mathrm{Polycon}(n) \to \mathbb{R}.$$

Via the embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $k \leq n$, we can regard $\mathrm{Polycon}(k)$ as a subcollection of $\mathrm{Polycon}(n)$ and thus we obtain a natural map

$$S_{k,n} : \mathbf{Val}(n) \to \mathbf{Val}(k)$$

We denote by $\mathbf{Val}^w(n)$ the vector subspace of $\mathbf{Val}(n)$ consisting of valuations μ homogeneous of degree (or weight) w, i.e.

$$\mu(tP) = t^w \mu(P), \ \ \forall t > 0, \ \ P \in \mathrm{Polycon}(n).$$

For every $k \leq n$ and every $\mu \in \mathbf{Val}(k)$ we define the *Radon transform* of μ to be

$$\mathcal{R}_{n,k}\mu : \mathrm{Polycon}(n) \to \mathbb{R}, \ \ \mathrm{Polycon}(n) \ni P \mapsto (\mathcal{R}_{n,k}\mu)(P) := \int_{\mathrm{Graff}(n,k)} \mu(P \cap V) d\lambda_k^n.$$

Proposition 3.30. *If $\mu \in \mathbf{Val}(k)$ then $\mathcal{R}_{n,k}\mu$ is a convex continuous, invariant valuation on* $\mathrm{Polycon}(n)$.

Proof. For every $\mu \in \mathbf{Val}(k)$, $V \in \mathrm{Graff}(n,k)$ and any $S_1, S_2 \in \mathrm{Polycon}(n)$ we have

$$V \cap (S_1 \cup S_2) = (V \cap S_1) \cup (V \cap S_2), \ \ (V \cap S_1) \cap (V \cap S_2) = V \cap (S_1 \cap S_2)$$

so that

$$\mu(\, V \cap (S_1 \cup S_2)\,) = \mu(V \cap S_1) + \mu(V \cap S_2) - \mu(V \cap (S_1 \cap S_2)).$$

Integrating the above equality with respect to $V \in \mathrm{Graff}(n,k)$ we deduce that

$$\mathcal{R}_{n,k}(S_1 \cup S_2) = \mathcal{R}_{n,k}(S_1) + \mathcal{R}_{n,k}(S_2) - \mathcal{R}_{n,k}(S_1 \cap S_2)$$

so that $\mathcal{R}_{n,k}\mu$ is a valuation on $\mathrm{Polycon}(n)$. The invariance of $\mathcal{R}_{n,k}$ follows from the invariance of μ and of the measure λ_k^n on $\mathrm{Graff}(n,k)$. Observe that if C_ν is a sequence of compact convex sets in $\mathbb{R}^n$ such that

$$C_\nu \to C \in \mathcal{K}^n$$

then

$$\lim_{\nu \to \infty} \mu(C_\nu \cap V) = \mu(C \cap V), \ \ \forall V \in \mathrm{Graff}(n,k).$$

We want to show that

$$\lim_{\nu \to \infty} \int_{\mathrm{Graff}(n,k)} \mu(C_\nu \cap V) d\lambda_k^n(V) = \int_{\mathrm{Graff}(n,k)} \mu(C \cap V) d\lambda_k^n(V)$$

by invoking the Dominated Convergence Theorem. We will produce an integrable function $f : \mathrm{Graff}(n,k) \to \mathbb{R}$ such that

$$\bigl|\, \mu(C_n \cap V) \,\bigr| \leq f(V), \ \ \forall n > 0, \ \ \forall V \in \mathrm{Graff}(n,k). \tag{3.6}$$

To this aim observe first that since $C_n \to C$ there exists $R > 0$ such that all the sets C_n and the set C are contained in the ball $\mathbf{B}_n(R)$ of radius R centered at the origin. Define

$$\mathrm{Graff}(n,k;R) := \bigl\{\, V \in \mathrm{Graff}(n,k) \mid V \cap \mathbf{B}_n(\mathbb{R}) \neq \emptyset \,\bigr\}.$$

$\mathrm{Graff}(n,k;R)$ is a compact subset of $\mathrm{Graff}(n,k)$ and thus it has finite volume. Now define

$$N = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\,\} \subset [0,1],$$

and

$$F : N \times \mathrm{Graff}(n,k;R) \to \mathbb{R}, \ \ F(r,V) := \bigl|\, \mu(C_{1/r} \cap V) \,\bigr|$$

where for uniformity we set $C := C_{1/0}$. Observe that $N \times \mathrm{Graff}(n,k;R)$ is a compact space and F is continuous. Set

$$M := \sup\bigl\{\, F(r,V) \mid (r,V) \in N \times \mathrm{Graff}(n,k;R) \,\bigr\}.$$

Then $M < \infty$. The function

$$f : \mathrm{Graff}(n,k) \to \mathbb{R}, \quad f(V) := \begin{cases} M & V \in \mathrm{Graff}(n,k;R) \\ 0 & V \notin \mathrm{Graff}(n,k;R) \end{cases}$$

satisfies the requirements of (3.6). $\square$

The resulting map $\mathcal{R}_{n,k} : \mathbf{Val}(k) \to \mathbf{Val}(n)$ is linear and it is called the *Radon transform*.

Proposition 3.31. *If $\mu \in \mathbf{Val}(k)$ is homogeneous of weight w, then $\mathcal{R}_{k+j,k}\mu$ is homogeneous of weight $w+j$, that is*

$$\mathcal{R}_{k+j,k}\big(\mathbf{Val}^w(k)\big) \subset \mathbf{Val}^{w+j}(k+j).$$

Proof. If $C \in \mathrm{Polycon}(k+j)$ and t is a positive real number then

$$(\mathcal{R}_{k+j,k}\mu)(tC) = \int_{\mathrm{Gr}(k+j,k)} \left(\int_{L^\perp} \mu(tC \cap (V+p)) d_{L^\perp} p \right) d\nu_k^{k+j}$$

We use the equality $tC \cap (V+p) = t\,(C \cap (V + t^{-1}p))$ to get

$$(\mathcal{R}_{k+j,k}\mu)(tC) = t^w \int_{\mathrm{Gr}(k+j,k)} \left(\int_{L^\perp} \mu(C \cap (V + t^{-1}p)) d_{L^\perp} p \right) d\nu_k^{k+j}$$

We make a change in variables $q = t^{-1}p$ in the interior integral so that $d_{L^\perp}p = t^j d_{L^\perp} q$ and

$$(\mathcal{R}_{k+j,k}\mu)(tC) = t^{w+j} \int_{\mathrm{Gr}(k+j,k)} \left(\int_{L^\perp} \mu(C \cap (V+q)) d_{L^\perp} q \right) d\nu_k^n = t^{w+j}\mathcal{R}_{k+j,k}\mu(C).$$

$\square$

So far we only know two valuations in $\mathbf{Val}(n)$, the Euler characteristic, and the Euclidean volume vol_n. We can now apply the Radon transform to these valuations and hopefully obtain new ones.

Example 3.32. Let $\mathrm{vol}_k \in \mathbf{Val}(k)$ denote the Euclidean k-dimensional volume in $\mathbb{R}^k$. Then for every $n > k$ and every compact convex subset $C \subset \mathbb{R}^n$ we have

$$(\mathcal{R}_{n,k}\,\mathrm{vol}_k)(C) = \int_{\mathrm{Graff}(n,k)} \mathrm{vol}_k(V \cap C) d\lambda_k^n(V)$$

$$= \int_{\mathrm{Gr}(n,k)} \left(\int_{L^\perp} \mathrm{vol}_k\big((C \cap (L+p) \big) d_{L^\perp} p \right) d\nu_k^n(L),$$

where $d_{L^\perp}p$ denotes the Euclidean volume element on the $(n-k)$-dimensional space $L^\perp$. The usual Fubini theorem applied to the decomposition $\mathbb{R}^n = L^\perp \times L$ implies that

$$\mathrm{vol}_n(C) = \mu_n(C) = \int_{\mathbb{R}^n} d_{\mathbb{R}^n} q = \int_{L^\perp} \mathrm{vol}_k\big((C \cap (L+p) \big) d_{L^\perp} p.$$

Hence

$$(\mathcal{R}_{n,k}\,\mathrm{vol}_k)(C) = \int_{\mathrm{Gr}(n,k)} \mathrm{vol}_n(C) d\nu_k^n(L) = \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{vol}_n(C).$$

In other words

$$\mathcal{R}_{n,k}\,\mathrm{vol}_k = \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{vol}_n\,.$$

Thus the Radon transform of the Euclidean volume produces the Euclidean volume in a higher dimensional space, rescaled by a universal multiplicative scalar. □

The above example is a bit disappointing since we have not produced an essentially new type of valuation by applying the Radon transform to the Euclidean volume. The situation is dramatically different when we play the same game with the Euler characteristic.

We are now ready to create elements of $\mathbf{Val}(n)$. For any positive integers k, n define

$$\mu_k^n := \mathcal{R}_{n,n-k}\mu_0 \in \mathbf{Val}(n),$$

where $\mu_0 \in \mathbf{Val}(k)$ denotes the Euler characteristic.

Observe that if C is a compact convex subset in $\mathbb{R}^{k+j}$, then for any $V \in \mathrm{Graff}(k+j,k)$ we have

$$\mu_0(V\cap C) = \begin{cases} 1 & \text{if } C\cap V \neq \emptyset \\ 0 & \text{if } C\cap V = \emptyset \end{cases}.$$

Thus the function

$$\mathrm{Graff}(k+j,k) \ni V \mapsto \mu_0(V\cap C)$$

is the indicator function of the set

$$\mathrm{Graff}(C,k) := \big\{ V \in \mathrm{Graff}(k+j,k);\;\; V\cap C \neq \emptyset \big\}.$$

We conclude that

$$\mu_j^{k+j}(C) = \left(\mathcal{R}_{k+j,k}\mu_0\right)(C) = \int_{\mathrm{Graff}(k+j,k)} I_{\mathrm{Graff}(C,k)}\, d\lambda_k^{k+j}$$
$$= \lambda_k^{k+j}\big(\,\mathrm{Graff}(k,C)\,\big).$$

Thus the quantity $\mu_j^{k+j}(C)$ "counts" how many k-planes intersect C. From this interpretation we obtain immediately the following result.

Theorem 3.33 (Sylvester)**.** *Suppose $K \subseteq L \subset \mathbb{R}^{k+j}$ are two compact convex subsets. Then the conditional probability that a k-plane which meets L also intersects K is equal to $\frac{\mu_j^{k+j}(K)}{\mu_j^{k+j}(L)}$.* □

We can give another interpretation of μ_j^n. Observe that if $C \in \mathcal{K}^n$ then

$$\mu_j^n(C) = \int_{\mathrm{Graff}(n,n-j)} \mu_0(C\cap V)\, d\lambda_{n-j}^n(V) = \int_{\mathrm{Gr}(n,n-j)} \left(\int_{L^\perp} \mu_0\big(\,C\cap L + p)\,\big)\, d_{L^\perp}p \right) d\nu_{n-j}^n(L).$$

Now observe that if $(C|L^\perp)$ denotes the orthogonal projection of C onto $L^\perp$ then

$$\mu_0\big(\,C\cap(L+p)\,\big) \neq 0 \Longleftrightarrow \mu_0\big(\,C\cap(L+p)\,\big) \Longleftrightarrow p \in (C|L^\perp).$$

An example of this fact in $\mathbb{R}^2$ can be seen in Figure 3. Hence

$$\int_{L^\perp} \mu_0\big(\,C\cap L+p)\,\big) d_{L^\perp}p = \mathrm{vol}_j(C|L^\perp)$$

and thus

$$\mu_j^n(C) = \int_{\mathrm{Gr}(n,n-j)} \mathrm{vol}_j(C|L^\perp) d\nu_{n-j}^n(L). \tag{3.7}$$

Thus μ_j^n is the "average value" of the volumes of the projections of C onto the j-dimensional subspaces of $\mathbb{R}^n$.

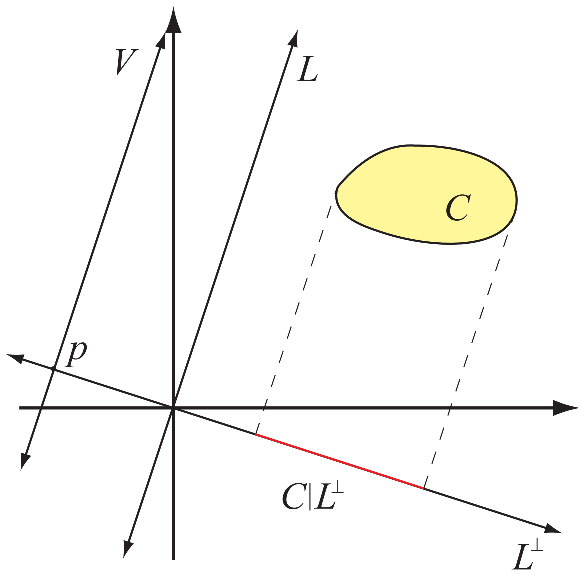

FIGURE 3. $C|L^\perp$ *for* V *in* $\mathrm{Graff}(2, 1)$.

A priori, some or maybe all of the valuations $\mu_j^n \in \mathbf{Val}^j(n) \subset \mathbf{Val}(n)$, $0 \leq j \leq n$ could be trivial. Note, however, that μ_n^n is vol_n. Furthermore, volume is intrinsic, so μ_n^n is in fact μ_n. We show that in fact all of the μ_j^n are nontrivial.

Proposition 3.34. *The valuations*

$$\mu_0 = \mu_0^n, \mu_1^n, \cdots, \mu_j^n, \cdots \mu_n \in \mathbf{Val}(n)$$

are linearly independent.

Proof. Let $\mathbf{B}_n(r)$ denote the closed ball of radius r centered at the origin of $\mathbb{R}^n$. We set $\mathbf{B}_n = \mathbf{B}_n(1)$. Since μ_j^n is homogeneous of degree j we have

$$\mu_j^n\big(\mathbf{B}_n(r)\big) = r^j \mu_j^n(\mathbf{B}_n) \stackrel{(3.7)}{=} r^j \int_{\mathrm{Gr}(n,n-j)} \mathrm{vol}_j(\mathbf{B}_n|L^\perp) d\nu_{n-j}^n(L) = r^j \int_{\mathrm{Gr}(n,n-j)} \mathrm{vol}_j(\mathbf{B}_j) d\nu_{n-j}^n(L).$$

Hence

$$\mu_j^n\big(\mathbf{B}_n(r)\big) = \omega_j r^j \int_{\mathrm{Gr}(n,n-j)} d\nu_{n-j}^n = \omega_j r^j \begin{bmatrix} n \\ n-j \end{bmatrix} = \omega_j r^j \begin{bmatrix} n \\ j \end{bmatrix}. \tag{3.8}$$

Observe that the above equality is also valid for $j = 0$.

Suppose that for some real constants $c_0, c_1, \cdots, c_n$ we have

$$\sum_{j=0}^n c_j \mu_j^n = 0.$$

Then

$$\sum_{j=0}^{n} c_j \mu_j^n\big(\mathbf{B}_n(r)\big) = 0, \quad \forall r > 0,$$

so that

$$\sum_{j=0}^{n} c_j \omega_j \begin{bmatrix} n \\ j \end{bmatrix} r^j = 0, \quad \forall r > 0.$$

This implies $c_j = 0$, $\forall j$.

□

The valuation μ_j^n is homogeneous of degree j and it induces a continuous, invariant, homogeneous valuation of degree w on the lattice $\mathrm{Pix}(n) \subset \mathrm{Polycon}(n)$. Observe that if $C_1 \subset C_2$ are two *compact convex* subsets then

$$\mu_j^n(C_1) \le \mu_j^n(C_2).$$

Since every n-dimensional parallelotope contains a ball of positive radius we deduce from (3.8) that $\mu_j^n(P) \neq 0$, for any n-dimensional parallelotope. Corollary 2.22 implies that there exists a constant $C = C_j^n$ such that

$$C_j^n \mu_j^n(P) = \mu_j(P), \quad \forall P \in \mathrm{Pix}(k+j). \tag{3.9}$$

We denote by $\widehat{\mu}_j^n$ the valuation

$$\widehat{\mu}_j^n := C_j^n \mu_j^n \in \mathbf{Val}^j(n). \tag{3.10}$$

Since μ_n^n is vol_n, as is $\widehat{\mu}_n^n$, we know that $C_n^n = 1$. Thus, $\widehat{\mu}_n^n = \widehat{\mu}_n = \mu_n = \mathrm{vol}_n$. We will prove in later sections that the sequence $(\widehat{\mu}_j^n)$ defines an *intrinsic* valuation and that in fact all the constants C_j^n are equal to 1.

Remark 3.35. Observe that

$$\mu_{n-1}^n(\mathbf{B}_n) = \omega_{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} = [n]\omega_{n-1} = \frac{n\omega_n}{2}$$

$$= \frac{\sigma_{n-1}}{2} = \frac{1}{2} \times \text{surface area of } \mathbf{B}_n.$$

This is a special case of a more general fact we discuss in the next subsection which will imply that the constants C_{n-1}^n in (3.9) are equal to 1. □

§3.7. **The Cauchy Projection Formula.** We want to investigate in some detail the valuations $\mu_j^{j+1} \in \mathbf{Val}(j+1)$. Suppose $C \in \mathcal{K}^{j+1}$ and $\ell \in \mathrm{Gr}(j+1,1)$. If we denote by $(C|\ell^\perp)$ the orthogonal projection of C onto the hyperplane $\ell^\perp$ then we obtain from (3.7)

$$\mu_j^{j+1}(C) = \int_{\mathrm{Gr}(j+1,1)} \mu_j(C|\ell^\perp)\, d\nu_1^{j+1}(\ell), \quad \forall C \in \mathcal{K}^{j+1}. \tag{3.11}$$

Loosely, speaking $\mu_j^{j+1}(C)$ is the average value of the "areas" of the shadows of C on the hyperplanes of $\mathbb{R}^{j+1}$. To clarify the meaning of this average we need the following elementary fact.

Lemma 3.36. *For any* $v \in \mathbb{S}^j \subset \mathbb{R}^{j+1}$,

$$\int_{\mathbf{S}^j} |u \cdot v| du = 2\omega_j \tag{3.12}$$

Proof. We recall from elementary calculus that an integral can be treated as a Riemann sum, i.e.,

$$\int_{\mathbf{S}^j} |u_i \cdot v| du \approx \sum_i |u_i \cdot v| S(A_i),$$

where the sphere $\mathbf{S}^j$ has been partitioned into many small portions (here denoted A_i), each containing the point u_i and possessing area $S(A_i)$.

Let $\widetilde{A}_i$ denote the orthogonal projection of A_i onto the hyperplane tangent to $\mathbf{S}^j$ at the point u_i. Denote by $\widetilde{A}_i|v^\perp$ the orthogonal projection of A_i onto the hyperplane $v^\perp$. Since $A_i \subset \mathbf{S}^j$, this projection is entirely into the disk $\mathbf{B}_{n-1}$ lying inside $v^\perp$. Since $\widetilde{A}_i$ lies in a hyperplane with unit normal $v^\perp$, its projected area is the product of its area in the hyperplane times the angle between the two planes, i.e. $S(\widetilde{A}_i|v^\perp) = |u_i \cdot v| S(\widetilde{A}_i)$.

For a fine enough partition of $\mathbf{S}^j$ we have that $S(A_i) \approx S(\widetilde{A}_i)$. Clearly then $|u_i \cdot v| S(A_i) \approx |u_i \cdot v| S(\widetilde{A}_i)$, and thus $S(A_i|v^\perp) \approx S(\widetilde{A}_i|v^\perp)$. Therefore,

$$\int_{\mathbf{S}^j} |u \cdot v| du \approx \sum_i S(A_i|v^\perp).$$

Because the collection of sets $\{A_i|v^\perp\}$ contains equivalent portions above and below the hyperplane $v^\perp$, it covers the unit, j-dimensional ball $\mathbf{B}_j$ once from above and once from below, we see that

$$\int_{\mathbf{S}^j} |u \cdot v| du \approx 2\omega_j.$$

The similarities in this proof all converge to equalities in the limit as the mesh of the partition A_i goes to zero. □

Theorem 3.37 (Cauchy's surface area formula)**.** *For every convex polytope* $K \subset \mathbb{R}^{j+1}$ *we denote by* $S(K)$ *its surface area. Then*

$$S(K) = \frac{1}{\omega_j} \int_{\mathbf{S}^j} \mu_j(K|u^\perp) du.$$

Proof. Let P have facets P_i with corresponding unit normal vectors v_i and surface areas α_i. For every unit vector u the projection $(P_i|u^\perp)$ onto the j-dimensional subspace orthogonal to u has j-dimensional volume

$$\mu_j(P_i|u^\perp) = \alpha_i |u \cdot v_i|.$$

For $u \in \mathbf{S}^j$, the region $(P|u^\perp)$ is covered completely by the projections of facets P_i,

$$(P|u^\perp) = \bigcup_i (P_i|u^\perp).$$

For almost every point p in the projection $(P|u^\perp)$ the line containing p and parallel to u intersects the boundary of P in two different points. (The set of points $p \in (P|u^\perp)$ for which this does happen is negligible, i.e. it has j-dimensional volume 0.) Thus, in the above decomposition of $(P|u^\perp)$ as a union of projection of facets, almost every point of $(P|u^\perp)$ is contained in exactly two such projections. Therefore

$$\mu_j(P|u^\perp) = \frac{1}{2}\sum_i \mu_j(P_i|u^\perp)$$

so that

$$\int_{\mathbf{S}^j} \mu_j(P|u^\perp)du = \int_{\mathbf{S}^j} \frac{1}{2}\sum_{i=1}^m \alpha_i|u\cdot v_i|du$$

$$= \sum_{i=1}^m \frac{\alpha_i}{2}\int_{\mathbf{S}^j} |u\cdot v_i|du \overset{(3.12)}{=} \sum_{i=1}^m \alpha_i\omega_j = \omega_j S(P).$$

□

Recall that for *parallelotopes* $P \in \mathrm{Par}(j+1)$,

$$\mu_j(P) = \frac{1}{2}S(P).$$

In this light, Cauchy's surface area formula becomes

$$\mu_j(P) = \frac{1}{2\omega_j}\int_{\mathbf{S}^j} \mu_j(P|u^\perp)du, \quad \forall P \in \mathrm{Par}(j+1).$$

We want to rewrite this as an integral over the Grassmannian $\mathrm{Gr}(j+1,1)$. Recall that we have a 2-to-1 map

$$\ell : \mathbf{S}^j \to \mathrm{Gr}(j+1,1).$$

An invariant measure μ_c on $\mathrm{Gr}(j+1,1)$ of total volume c corresponds to an invariant measure $\tilde{\mu}_c$ on $\mathbf{S}^j$ of total volume $2c$. If σ denotes the invariant measure on $\mathbf{S}^j$ defined by the area element on the sphere of radius 1 then

$$\sigma(\mathbf{S}^j) = \sigma_j, \quad \tilde{\mu}_c = \frac{2c}{\sigma_j}\sigma$$

Any function $f : \mathrm{Gr}(j+1,1) \to \mathbb{R}$ defines a function $\ell^* f := f \circ \ell : \mathbf{S}^j \to \mathbb{R}$ called the *pullback* of f via ℓ. Conversely, any even function g on the sphere is the pullback of some function $\bar{g}$ on $\mathrm{Gr}(j+1,1)$. Moreover

$$\int_{\mathrm{Gr}(j+1,1)} f d\mu_c = \frac{1}{2}\int_{\mathbf{S}^j} \ell^* f d\tilde{\mu}_c = \frac{c}{\sigma_j}\int_{\mathbf{S}^j} \ell^* f d\sigma.$$

This shows that

$$\int_{\mathbf{S}^j} \ell^* f d\sigma = \frac{\sigma_j}{c}\int_{\mathrm{Gr}(j+1,1)} f d\mu_c.$$

The measure ν_1^{j+1} has total volume $c = [j+1]$ so that

$$\mu_j(P) = \frac{1}{2\omega_j}\int_{\mathbf{S}^j} \mu_j(P|u^\perp)du = \frac{\sigma_j}{2[j+1]\omega_j}\int_{\mathrm{Gr}(j+1,1)} \mu_j(P|L^\perp)d\nu_1^{j+1}(L)$$

Using (3.11) we deduce

$$\mu_j(P) = \frac{\sigma_j}{2[j+1]\omega_j}\mu_j^{j+1}(P)$$

for every parallelotope $P \subset \mathbb{R}^{j+1}$. Recalling that

$$\sigma_j = (j+1)\omega_{j+1}, \quad [j+1] = \frac{(j+1)\omega_{j+1}}{2\omega_j}$$

we conclude

$$\frac{\sigma_j}{2[j+1]\omega_j} = 1$$

so that finally

$$\mu_j(P) = \mu_j^{j+1}(P), \quad \forall P \in \operatorname{Par}(j+1).$$

4. The Characterization Theorem

We are now ready to completely characterize $\mathbf{Val}(n)$.

§4.1. **The Characterization of Simple Valuations.** Before we can state and prove the characterization theorem for simple valuations, we need to recall a few things. E_n denotes the group of rigid motions of $\mathbb{R}^n$, i.e. the subgroup of affine transformations of $\mathbb{R}^n$ generated by translations and rotations. Two subsets $S_0, S_1 \subset \mathbb{R}^n$ are called *congruent* if there exists $\phi \in E_n$ such that

$$\phi(S_0) = S_1.$$

A valuation $\mu : \mathrm{Polycon}(n) \to \mathbb{R}$ is called *simple* if

$$\mu(S) = 0, \quad \text{for every } S \in Polycon(n) \text{ such that } \dim S < n.$$

The *Minkowski Sum* of two compact convex sets, K, and L is given by

$$K + L = \{ x + y \mid x \in K, y \in L \}.$$

We call a *zonotope* a finite Minkowski sum of straight line segments, and we call a *zonoid* a convex set Y that can be approximated in the Hausdorff metric by a convergent sequence of zonotopes.

If a compact convex set is symmetric about the origin, the we call it a *centered* set. We denote the set of centered sets in $\mathcal{K}^n$ by $\mathcal{K}^n_c$.

The proof of the characterization theorem relies in a crucial way on the following technical result.

Lemma 4.1. *Let $K \in \mathcal{K}^n_c$. Suppose that the support function of K is smooth. Then there exist zonoids Y_1, Y_2 such that*

$$K + Y_2 = Y_1.$$

□

Idea of proof. We begin by observing that the support function of a centered line segment in $\mathbb{R}^n$ with endpoints $u, -u$ is given by

$$h_u : \mathbf{S}^{n-1} \to \mathbb{R}, \quad h(x) := |\langle u, x \rangle|.$$

Thus, any function $g : \mathbf{S}^{n-1} \to \mathbb{R}$ which is a linear combinations of h_u's with positive coefficients is the support function of a centered zonotope. In particular any uniform limit of such linear combinations will be the support function of a zonoid. For example, a function

$$g : \mathbf{S}^{n-1} \to \mathbb{R}$$

of the form

$$g(x) = \int_{\mathbf{S}^{n-1}} |\langle u, x \rangle| f(u) dS_u,$$

with $f : \mathbf{S}^{n-1} \to [0, \infty)$ a continuous, symmetric function, is the support function of a zonoid.

Let us denote by $C_{even}(\mathbf{S}^{n-1})$ the space of continuous, symmetric (even) functions $\mathbf{S}^{n-1} \to \mathbb{R}$. We obtain a linear map

$$\mathcal{C} : C_{even}(\mathbf{S}^{n-1}) \to C_{even}(\mathbf{S}^{n-1}), \ \ (\mathcal{C}f)(x) := \int_{\mathbf{S}^{n-1}} |\langle u, x\rangle| f(u) dS_u$$

called the *cosine transform*. Thus we can rephrase the last remark by saying that the cosine transform of a *nonnegative* continuous, even function on $\mathbf{S}^{n-1}$ is the support function of a zonoid.

Note that f is continuous, even, then we can write is as the difference of two continuous, even, *nonnegative* functions

$$f = f_+ - f_-, \ \ f_+ = \frac{1}{2}(f + |f|), \ \ f_- = \frac{1}{2}(|f| - f).$$

Thus if h is the support function of some $C \in \mathcal{K}^n_c$ such that

$$h = \mathcal{C}f \Longrightarrow h + \mathcal{C}f_- = \mathcal{C}f_+$$

Note that $\mathcal{C}f_-$ is the support function of a zonoid Y_1 and $\mathcal{C}f_+$ is the support function of a zonoid Y_1 and thus

$$h + \mathcal{C}f_- = \mathcal{C}f_+ \Longleftrightarrow C + Y_1 = Y_2.$$

We deduce that any continuous function which is in the image of the cosine transform is the support function of a set $C \in \mathcal{K}^n$ satisfying the conclusions of the lemma.

The punch line is now that *any* smooth, even function $f : \mathbf{S}^{n-1} \to \mathbb{R}$ *is* the cosine transform of s smooth, even function on the sphere.

This existence statement is a nontrivial result, and its proof is based on a very detailed knowledge of the action of the cosine transform on the harmonic polynomials on $\mathbf{S}^{n-1}$. For a proof along these lines we refer to [**Gr**, Chap. 3]. For an alternate approach which reduces the problem to a similar statement about Fourier transforms we refer to [**Gon**, Prop. 6.3]. □

Remark 4.2. The connection between the classification of valuations and the spectral properties of the cosine transform is philosophically the key reason why the classification turns out to be simple. The essence of the classification is invariant theoretic, and is roughly says that the invariance under the group of motions dramatically cuts down the number of possible choices. □

Theorem 4.3. *Suppose that $\mu \in \mathbf{Val}(n)$ is a simple valuation. If μ is* even, *that is*

$$\mu(K) = \mu(-K), \ \ \forall K \in \mathcal{K}^n,$$

then

$$\mu([0,1]^n) = 0 \Longleftrightarrow \mu \text{ is identically zero on } \mathcal{K}^n.$$

Proof. Clearly only the implication $\Longrightarrow$ is nontrivial. To prove it we employ a simple strategy. By cleverly cutting and pasting and taking limits we produce more and more sets $S \in \mathrm{Polycon}(n)$ such that $\mu(S) = 0$ until we get what we want. We will argue by induction on the dimension n of the ambient space.

The result is true for $n = 1$ because in this case $\mathcal{K}^1 = \mathrm{Par}(1)$ and we can invoke the volume theorem for $\mathrm{Pix}(1)$, Theorem 2.19.

So, now suppose that $n > 1$ and the theorem holds for valuations on $\mathcal{K}^{n-1}$. Note that for every positive integer k the cube $[0,1]$ decomposes in k^n cubes congruent to $\left[0, \frac{1}{k}\right]$ and overlapping on sets of dimension $< n$. We deduce that

$$\mu([0,1/k]^n) = 0.$$

Since any box whose edges have rational length decomposes into cubes congruent to $\left[0, \frac{1}{k}\right]^n$ for some positive integer k we deduce that μ vanishes on such boxes. By continuity we deduce that it vanishes on all boxes $P \in \mathrm{Par}(n)$. Using the rotation invariance we conclude that rotation invariance, $\mu(C) = 0$ for all boxes C with positive sides parallel to some other orthonormal axes.

We now consider *orthogonal cylinders* with convex bases, i.e. sets congruent to convex sets of the form $C \times [0,r] \in \mathcal{K}^n$, $C \in \mathcal{K}^{n-1}$. For every real numbers $a < b$ define a valuation $\tau = \tau_{a,b}$ on $\mathcal{K}^{n-1}$ by

$$\tau(K) = \mu(K \times [a,b]),$$

for all $K \in \mathcal{K}^{n-1}$. Note that $[0,1]^{n-1} \times [a,b] \in \mathrm{Par}(n)$ so that

$$\tau([0,1]^{n-1}) = \mu([0,1]^{n-1} \times [a,b]) = 0.$$

Then τ satisfies the inductive hypotheses, so $\tau = 0$. Hence μ vanishes on all orthogonal cylinders with convex basis.

Now suppose that M is a *prism*, with congruent top and bottom faces parallel to the hyperplane $x_n = 0$, but whose cylindrical boundary is not orthogonal to $x_n = 0$ but rather meets it at some constant angle. See the top of Figure 4

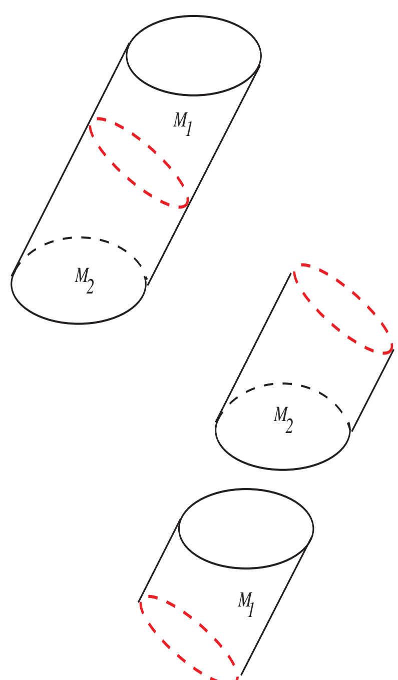

FIGURE 4. *Cutting and pasting a prism.*

We now cut M into two pieces M_1, M_2 by a hyperplane orthogonal to the cylindrical boundary of M. Using translation invariance, slide M_1 and M_2 around and glue them together along the original top and bottom faces. (See Figure 4). We then have a right cylinder C. Thus,

$$\mu(M) = \mu(M_1) + \mu(M_2) = \mu(C) = 0.$$

Note that we must actually be careful with this cutting and repasting. It is possible that M is too "fat", preventing us from slicing M with a hyperplane orthogonal to the cylindrical axes and fully contained in each of them. If this problem occurs, we can easily remedy it by subdividing the top and bottom of M into sufficiently small convex pieces. Using the simplicity of μ, we can then consider each separately.

Now let P be a convex polytope having facets $P_1, \ldots, P_m$, and corresponding outward unit normal vectors $u_1, \ldots, u_m$. Let $v \in \mathbb{R}^n$ and let $\bar{v}$ denote the straight line segment connecting the point v to the origin. Without loss of generality, suppose that $P_1, \ldots, P_j$ are exactly those facets of P such that $\langle u_i, v\rangle > 0$ for all $1 \leq i \leq j$. We can thus express the Minkowski sum $P + \bar{v}$ as

$$P + \bar{v} = P \cup \left(\bigcup_{i=1}^{j} (P_i + \bar{v}) \right),$$

where each term in the above union is either disjoint from the others or intersects in dimension less than n (see Figure 5).

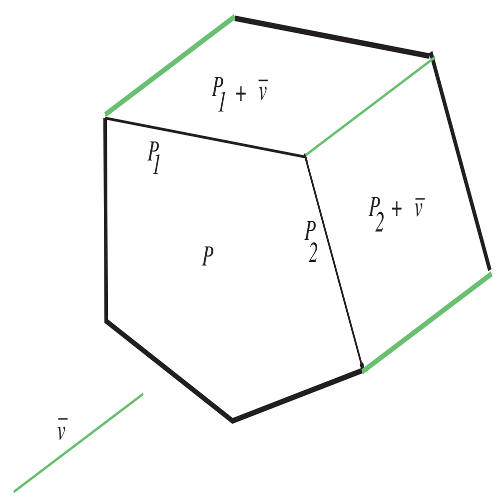

FIGURE 5. *Smearing a convex polytope.*

This simply results from the fact that $P + \bar{v}$ is the 'smearing' of P in the direction of v. Hence, since μ is simple,

$$\mu(P + \bar{v}) = \mu(P) + \sum_{i=1}^{j} \mu(P_i + \bar{v}).$$

But now each term $P_i + \bar{v}$ is the smearing of the facet P_i in the direction of v, making $P_i + \bar{v}$ a prism, so that

$$\mu(P + \bar{v}) = \mu(P),$$

for all convex polytopes P and line segments $\bar{v}$. By translation invariance $\bar{v}$ could be *any* segment in the space.

We deduce iteratively that for all convex polytopes P and all zonotopes Z we have

$$\mu(Z) = 0 \quad \text{and} \quad \mu(P + Z) = \mu(P).$$

By continuity,

$$\mu(Y) = 0 \quad \text{and} \quad \mu(K + Y) = \mu(K),$$

for all $K \in \mathcal{K}^n$ and all zonoids Y.

Now suppose that $K \in \mathcal{K}_c^n$ and has a smooth support function. Then by our lemma, there are zonoids Y_1, Y_2 such that $K + Y_1 = Y_2$. Thus, we now have that

$$\mu(K) = \mu(K + Y_2) = \mu(Y_1) = 0.$$

Since any centered convex body can be approximated by a sequence of centered convex sets with smooth support functions, by continuity of μ, μ is zero on all centered convex, compact sets.

Now let Δ be an n-dimensional simplex, with one vertex at the origin. Let $u_1, \dots, u_n$ denote the other vertices of Δ, and let P be the parallelepiped spanned by the vectors $u_1, \dots, u_n$. Let $v = u_1 + \cdots + u_n$. Let ξ_1 be the hyperplane passing through the points $u_1, \dots, u_n$ and let ξ_2 be the hyperplane passing through the points $v - u_1, \dots, v - u_n$. Finally, denote by Q the set of all points of P lying between the hyperplanes ξ_1 and ξ_2.

We now write

$$P = \Delta \cup Q \cup (-\Delta + v),$$

where each term in the union intersects any other in dimension less than n. P and Q are centered, so

$$0 = \mu(P) = \mu(\Delta) + \mu(Q) + \mu(-\Delta + v) = \mu(\Delta) + \mu(-\Delta).$$

Thus, $\mu(\Delta) = -\mu(-\Delta)$. Yet by assumption $\mu(\Delta) = \mu(-\Delta)$. Thus, $\mu(\Delta) = 0$.

Now since any convex polytope can be expressed as the finite union of simplices each of which intersects with any other in dimension less than n, we have that $\mu(P) = 0$ for all convex polytopes P. Since convex polytopes are dense in $\mathcal{K}^n$, we have that μ is zero on $\mathcal{K}^n$, which is what we wanted. □

We now immediately get the following result.

Theorem 4.4. *Suppose that μ is a continuous simple valuation on $\mathcal{K}^n$ that is translation and rotation invariant. Then there exists some $c \in \mathbb{R}$ such that $\mu(K) + \mu(-K) = c\mu_n(K)$ for all $K \in \mathcal{K}^n$.*

Proof. For $K \in \mathcal{K}^n$, define

$$\eta(K) = \mu(K) + \mu(-K) - 2\mu([0,1]^n)\mu_n(K).$$

Then η satisfies the conditions of the previous theorem, so that η is zero on $\mathcal{K}^n$. Thus,

$$\mu(K) + \mu(-K) = c\mu_n(K),$$

where $c = 2\mu([0,1]^n)$. □

§4.2. **The Volume Theorem.**

Theorem 4.5 (Sah)**.** *Let Δ be a n-dimensional simplex. There exist convex polytopes $P_1, \ldots, P_m$ such that $\Delta = P_1 \cup \cdots \cup P_m$ where each term of the union intersects another in dimension at most $n-1$ and where each of the polytopes P_i is symmetric under a reflection across a hyperplane, i.e. for each P_i, the exists a hyperplane such that P_i is symmetric when reflected across it.*

Proof. Let $x_0, \ldots, x_n$ be the vertices of Δ and let Δ_i be the facet of Δ opposite to x_i. Let z be the center of the inscribed sphere of Δ and let z_i be the foot of the perpendicular line from z to the facet Δ_i. For all $i < j$, let $A_{i,j}$ denote the convex hull of z, z_i, z_j and the face $\Delta_i \cap \Delta_j$ (see Figure 6).

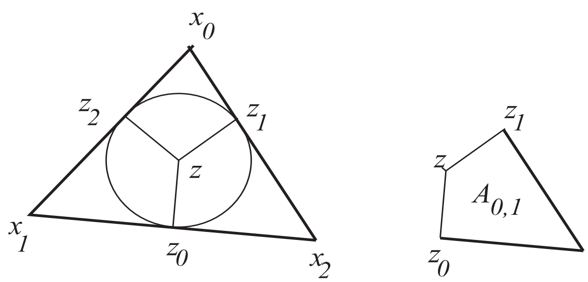

FIGURE 6. *Cutting a simplex into symmetric polytopes.*

Then

$$\Delta = \bigcup_{0 \le i < j \le n} A_{i,j},$$

where the distinct terms $A_{i,j}$ of this union, by construction, intersect in at most dimension $n-1$. Each $A_{i,j}$ is symmetric under reflection across the $n-1$ hyperplane determined by z and the face $\Delta_i \cap \Delta_j$. We can then relabel the $A_{i,j}$ as $P_1, \ldots, P_m$ and $\Delta = P_1 \cup \cdots \cup P_m$ as desired. □

Theorem 4.6 (Volume Theorem for $\mathrm{Polycon}(n)$)**.** *Suppose that μ is a continuous rigid motion invariant simple valuation on $\mathcal{K}^n$. Then there exists $c \in \mathbb{R}$ such that $\mu(K) = c\mu_n(K)$, for all $K \in \mathcal{K}^n$. Note that we can extend continuous valuations on $\mathcal{K}^n$ to continuous valuations on $\mathrm{Polycon}(n)$, so the theorem also holds replacing $\mathcal{K}^n$ with $\mathrm{Polycon}(n)$.*

Proof. Since μ is translation invariant and simple, by Theorem 4.4, there exists $a \in \mathbb{R}$ such that $\mu(K) + \mu(-K) = a\mu_n(K)$ for all $K \in \mathcal{K}^n$. Let Δ be a n-simplex in $\mathbb{R}^n$. Then $\mu(\Delta) + \mu(-\Delta) = a\mu_n(\Delta)$.

Suppose n is even, meaning the dimension of $\mathbb{R}^n$ is even. Then Δ and $-\Delta$ differ by a rotation. This is clearly true in $\mathbb{R}^2$. We can then rotate Δ to $-\Delta$ in each "orthogonal" plane of $\mathbb{R}^n$, meaning Δ and $-\Delta$ differ by the composition of these rotations, i.e. a rotation. Then $\mu(\Delta) = \mu(-\Delta) = \frac{a}{2}\mu_n(\Delta)$.

Now suppose that n is odd. By Theorem 4.5, there exist polytopes $P_1, \ldots, P_m$ such that $\Delta = P_1 \cup \cdots \cup P_m$ and $\dim P_i \cap P_j \le n-1$ and each P_i is symmetric under a reflection across a hyperplane. Thus, we can transform P_i into $-P_i$ by a series of rotations and then reflection across the hyperplane. Then, $\mu(P_i) = \mu(-P_i)$ and

$$\mu(-\Delta) = \sum_{i=1}^{m} \mu(-P_i) = \sum_{i=1}^{m} \mu(P_i) = \mu(\Delta)$$

Then for all n-simplices Δ, $\mu(\Delta) = \frac{a}{2}\mu_n(\Delta)$.

Let $c = \frac{a}{2}$ and suppose P is a convex polytope in $\mathbb{R}^n$. Then P is a finite union of simplices, i.e. $P = \Delta_1 \cup \cdots \cup \Delta_m$ such that $\dim \Delta_i \cap \Delta_j < n$. Then

$$\begin{aligned} \mu(P) &= \mu(\Delta_1) + \cdots + \mu(\Delta_m) \\ &= c\mu_n(\Delta_1) + \cdots + c\mu_n(\Delta_m) \\ &= c\mu_n(P) \end{aligned}$$

The set of convex polytopes is dense in $\mathcal{K}^n$, so since μ is continuous, $\mu(K) = c\mu_n(K)$ for all $K \in \mathcal{K}^n$. □

§4.3. **Intrinsic Valuations.** We would like to show that for each $k \geq 0$ the valuations $\widehat{\mu}_k^n$ determine an *intrinsic valuation*, meaning that for all $N \geq n$ and $P \in \mathrm{Polycon}(n)$, $\widehat{\mu}_k^n(P) = \widehat{\mu}_k^N(P)$. For uniformity, we set

$$\widehat{\mu}_k^n = 0, \quad \forall k > n.$$

Theorem 4.7. *For every $k \geq 0$ the sequence*

$$(\widehat{\mu}_k^n) \in \prod_{n \geq 0} \mathbf{Val}(n)$$

defines an intrinsic valuation, i.e., an element of

$$\widehat{\mu}_k \in \mathbf{Val}(\infty) = \mathrm{limproj}_n \, \mathbf{Val}(n).$$

Proof. First of all let us fix k. Since there is no danger of confusion we will write $\widehat{\mu}^n$ instead of $\widehat{\mu}_k^n$. Consider the statement

$$\widehat{\mu}^n(P) = \widehat{\mu}^\ell(P), \quad \forall P \in \mathrm{Polycon}(\ell). \qquad (\mathbf{P}_{n,\ell})$$

Note that $l \leq n$. We want to prove by induction over n that the statement

$$P_{n,\ell} \text{ for any } \ell \leq n \qquad (\mathbf{S}_n)$$

is true for any n. The statement $\mathbf{S}_0$ is trivially true. We will prove that

$$\mathbf{S}_0, \cdots, \mathbf{S}_{n-1} \Longrightarrow \mathbf{S}_n.$$

Thus, we know that $P_{m,\ell}$ is true for every $\ell \leq m < n$, and we want to prove that

$$P_{n,\ell} \quad \text{is true for any } \ell. \qquad (\mathbf{T}_\ell)$$

To prove T_ℓ we argue by induction on ℓ. (n is fixed.)

The result is obviously true for $\ell = 0, 1$ since in these case $\mathrm{Pix} = \mathrm{Polycon}$. Thus, we assume that $P_{n,\ell}$ is true and we will show that $P_{n,\ell+1}$ is true. In other words, we know that

$$\widehat{\mu}^n(P) = \widehat{\mu}^\ell(P), \quad P \in \mathrm{Polycon}(\ell) \qquad (4.1)$$

and we want to prove that

$$\widehat{\mu}^{n}(P) = \widehat{\mu}^{\ell+1}(P), \quad \forall P \in \mathrm{Polycon}(\ell+1).$$

Clearly the last statement is trivially true if $\ell + 1 = n$ so we may assume $\ell + 1 < n$.

We denote by ν the restriction of $\widehat{\mu}^{n}$ to $\mathrm{Polycon}(\ell+1)$. Then ν restricts to $\widehat{\mu}^{\ell}$ on $\mathrm{Polycon}(\ell)$ and $\widehat{\mu}^{n}$ restricts to $\widehat{\mu}^{\ell}$ by (4.1).

Since $\ell + 1 < n$ we deduce from $\mathbf{S}_{\ell+1}$ that $\widehat{\mu}^{\ell+1}$ restricts to $\widehat{\mu}^{\ell}$ on $\mathrm{Polycon}(\ell)$. Then $\nu - \mu^{\ell+1}$ vanishes on $\mathrm{Polycon}(\ell)$, meaning it is a continuous invariant simple valuation on $\mathrm{Polycon}(\ell+1)$. Theorem 4.6 implies that there exists $c \in \mathbb{R}$ such that $\nu - \widehat{\mu}^{\ell+1} = c\widehat{\mu}^{\ell+1}_{\ell+1}$ on $\mathrm{Polycon}(\ell+1)$.

On the other hand, $\nu = \widehat{\mu}^{l+1}$ on $\mathrm{Pix}(l+1)$, meaning $c = 0$ and thus $\nu = \widehat{\mu}^{\ell+1}$. □

Based on this result, the superscript of $\widehat{\mu}^{n}_{k}$ does not matter. Therefore, we define

$$\widehat{\mu}_{k} := \widehat{\mu}^{n}_{k}.$$

At this point, we are able to characterize the continuous, invariant valuations on $\mathrm{Polycon}(n)$, as we did for $\mathrm{Pix}(n)$ in Theorem 2.20.

Theorem 4.8 (Hadwiger's Characterization Theorem)**.** *The valuations* $\widehat{\mu}_0, \widehat{\mu}_1, \ldots, \widehat{\mu}_n$ *form a basis for the vector space* $\mathbf{Val}(n)$.

Proof. Let $\mu \in \mathbf{Val}(n)$ and let H be a hyperplane in $\mathbb{R}^n$. The restriction of μ to H is then a continuous invariant valuation on H. Note that the choice of H does not matter since μ is rigid motion invariant and all hyperplanes can be arrived at through rigid motions of a single hyperplane. Recall that $\mathrm{Polycon}(1) = \mathrm{Pix}(1)$, and by Theorem 2.20, the statement is true for $n = 1$. We take this as our base case and proceed by induction.

For every polyconvex set A in H, we assume that

$$\mu(A) = \sum_{i=0}^{n-1} c_i \widehat{\mu}_i(A)$$

Thus,

$$\mu - \sum_{i=0}^{n-1} c_i \widehat{\mu}_i$$

is a simple valuation in $\mathbf{Val}(n)$. Then, by Theorem 4.6,

$$\mu - \sum_{i=0}^{n-1} c_i \widehat{\mu}_i = c_n \widehat{\mu}_n$$

We move the sum to the other side and

$$\mu = \sum_{i=0}^{n} c_i \widehat{\mu}_i$$

□

In a definition analogous to that on $\mathrm{Pix}(n)$, a valuation μ on $\mathrm{Polycon}(n)$ is homogenous of degree k if for $\alpha \geq 0$ and all $K \in \mathrm{Polycon}(n)$, $\mu(\alpha K) = \alpha^k \mu(K)$.

Corollary 4.9. *Let* $\mu \in \mathbf{Val}(n)$ *be homogenous of degree* k*. Then there exists* $c \in \mathbb{R}$ *such that* $\mu(K) = c\widehat{\mu}_k(K)$ *for all* $K \in \mathrm{Polycon}(n)$.

Proof. By Theorem 4.8, we know that there exist $c_0, \cdots, c_n \in \mathbb{R}$ such that

$$\mu = \sum_{i=0}^{n} c_i \widehat{\mu}_i$$

Suppose $P = [0,1]^n$, the unit cube in $\mathbb{R}^n$. Fix $\alpha \geq 0$. Then,

$$\mu(\alpha P) = \sum_{i=0}^{n} c_i \widehat{\mu}_i(\alpha P) = \sum_{i=0}^{n} \alpha^i c_i \widehat{\mu}_i(P) = \sum_{i=0}^{n} \binom{n}{i} c_i \alpha^i.$$

$\widehat{\mu}_i(P) = \mu_i(P) = \binom{n}{i}$ since for $P \in \mathrm{Par}(n)$, μ_i is the i-th elementary symmetric function. At the same time,

$$\mu(\alpha P) = \alpha^k \mu(P) = \sum_{i=0}^{n} c_i \widehat{\mu}_i(P) \alpha^k = \sum_{i=0}^{n} \binom{n}{i} c_i \alpha^k.$$

We compare the coefficients c_i in the two sums and conclude that $c_j = 0$ for $i \neq k$. Thus, $\mu = c_k \widehat{\mu}_k$. □

5. **Applications**

It is payoff time! Having thus proven Hadwiger's Characterization Theorem, we now seek to use it in extracting as much interesting geometric information as possible. Let us recall that

$$\mu_k^n = \mathcal{R}_{n,n-k}\mu_0 \ \text{ and } \ \widehat{\mu}_k^n = C_k^n \mu_k^n,$$

where the normalization constants C_k^n are chosen so that

$$\widehat{\mu}_k^n([0,1]^n) = \mu_k([0,1]^n) = \binom{n}{k}.$$

Above, μ_k is the intrinsic valuation on the lattice of pixelations defined in Section 2. We have seen in the last section that the sequence $\{\widehat{\mu}_k^n\}_{n\geq k}$ defines an intrinsic valuation and thus we will write $\widehat{\mu}_k$ instead of $\widehat{\mu}_k^n$

§5.1. **The Tube Formula.**

Let $K, L \in \mathcal{K}^n$, and let $\alpha \geq 0$. We have the Minkowski sum

$$K + \alpha L = \{x + \alpha y \mid x \in K \text{ and } y \in L\}$$

Proposition 5.1 (Smearing Formula)**.** *Let $\overline{u}$ denote the straight line segment connecting u and the origin o. For $K \in \mathcal{K}^n$ and any unit vector $u \in \mathbb{R}^n$,*

$$\mu_n(K + \epsilon\overline{u}) = \mu_n(K) + \epsilon\mu_{n-1}(K|u^\perp)$$

Proof. Let $L = K + \epsilon\overline{u}$. We will compute the volume of L by integrating the lengths of one-dimensional slices of L with lines ℓ_x parallel through u passing through points $x \in u^\perp$, that is

$$\mu_n(L) = \int_{u^\perp} \mu_1(L \cap \ell_x)dx.$$

Since $\mu_1(l \cap \ell_x) = \mu_1(K \cap \ell_x) + \epsilon$ for all $x \in K|u^\perp$ and zero for $x \notin K|u^\perp$, we have that

$$\mu_n(L) = \int_{u^\perp} \mu_1(L \cap \ell_x)dx = \int_{K|u^\perp} (\mu_1(K \cap \ell_x) + \epsilon)dx = \mu_n(K) + \epsilon\mu_{n-1}(K|u^\perp)$$

□

Let C_n denote the n-dimensional unit cube. Recall that for $0 \leq i \leq n$,

$$\mu_i(C_n) = \binom{n}{i}.$$

Proposition 5.2. *For $\epsilon \geq 0$,*

$$\mu_n(C_n + \epsilon\mathbf{B}_n) = \sum_{i=0}^{n} \mu_i(C_n)\omega_{n-i}\epsilon^{n-i} = \sum_{i=0}^{n} \binom{n}{i}\omega_{n-i}\epsilon^{n-i} = \sum_{i=0}^{n} \omega_{n-i}\widehat{\mu}_i^n(C_n)\epsilon^{n-i}.$$

Proof. Let $u_1, \ldots, u_n$ denote the standard orthonormal basis for $\mathbb{R}^n$, and $\overline{u}_i$ the line segment connecting u_i to the origin. By Proposition 5.1 we have

$$\mu_n(\mathbf{B}_n + r\overline{u}_1) = \omega_n + r\omega_{n-1} = \sum_{i=0}^{n} \binom{1}{i} \omega_{n-i} r^i$$

for all $n \geq 1$.

Having proven the base case, we proceed by induction. Suppose that

$$\mu_n(\mathbf{B}_n + r\overline{u}_1 + \cdots + r\overline{u}_k) = \sum_{i=0}^{k} \binom{k}{i} \omega_{n-i} r^i,$$

for some $1 \leq k < n$. Then

$$\mu_n(\mathbf{B}_n+r\overline{u}_1+\cdots+r\overline{u}_{k+1}) = \mu_n(\mathbf{B}_n+r\overline{u}_1+\cdots+r\overline{u}_k)+r\mu_{n-1}((\mathbf{B}_{n-1}+r\overline{u}_1+\cdots+r\overline{u}_k)|u_{k+1}^{\perp})$$

$$= \sum_{i=0}^{k} \binom{k}{i} \omega_{n-i} r^i + r\mu_{n-1}(\mathbf{B}_{n-1}+r\overline{u}_1+\cdots+\overline{u}_k) = \sum_{i=0}^{k} \binom{k}{i} \omega_{n-i} r^i + \sum_{i=0}^{k} \binom{k}{i} \omega_{n-i-1} r^{i+1}$$

$$= \sum_{i=0}^{k+1} \left(\binom{k}{i} + \binom{k}{i-1} \right) \omega_{n-i} r^i = \sum_{i=0}^{k+1} \binom{k+1}{i} \omega_{n-i} r^i.$$

Thus, by induction we have that

$$\mu_n(\mathbf{B}_n + r\overline{u}_1 + \cdots + r\overline{u}_n) = \sum_{i=0}^{n} \binom{n}{i} \omega_{n-i} r^i$$

Noting that μ_n is homogeneous of degree n, we have that

$$\mu_n(\mathbf{B}_n+r\overline{u}_1+\cdots+r\overline{u}_n) = \mu_n(\mathbf{B}_n+rC_n) = \mu_n\left(r\left(\frac{1}{r}\mathbf{B}_n + C_n\right)\right) = r^n \mu_n\left(\frac{1}{r}\mathbf{B}_n + C_n\right).$$

Letting $\epsilon = \frac{1}{r}$, the previous two inequalities give us

$$\mu_n(C_n + \epsilon \mathbf{B}_n) = \epsilon^n \sum_{i=0}^{n} \binom{n}{i} \omega_{n-i} \frac{1}{\epsilon}^i = \sum_{i=0}^{n} \binom{n}{i} \omega_{n-i} \epsilon^{n-i}$$

□

We can now prove the following remarkable result, first proved by Jakob Steiner in dimensions ≤ 3 in the 19th century.

Theorem 5.3 (Tube Formula)**.** *For $K \in \mathcal{K}^n$ and $\epsilon \geq 0$,*

$$\mu_n(K + \epsilon \mathbf{B}_n) = \sum_{i=0}^{n} \widehat{\mu}_i(K) \omega_{n-i} \epsilon^{n-i}. \tag{5.1}$$

Proof. Let $\eta(K) = \mu_n(K + \mathbf{B}_n)$, for $K \in \mathcal{K}^n$. Because $\mathbf{B}_n \in \mathcal{K}^n$, for $K, L \in \mathcal{K}^n$ we have that

$$(K \cup L) + \mathbf{B}_n = (K + \mathbf{B}_n) \cup (L + \mathbf{B}_n),$$

and clearly

$$(K + \mathbf{B}_n) \cap (L + \mathbf{B}_n) = (K \cap L) + \mathbf{B}_n,$$

so we have

$$\begin{aligned}\eta(K \cup L) = \mu_n\Big((K \cup L) + \mathbf{B}_n \Big) &= \mu_n\Big((K + \mathbf{B}_n) \cup (L + \mathbf{B}_n) \Big) \\ = \mu_n(K + \mathbf{B}_n) + \mu_n(L + \mathbf{B}_n) &- \mu_n\Big((K + \mathbf{B}_n) \cap (L + \mathbf{B}_n) \Big) \\ &= \eta(K) + \eta(L) - \eta(K \cap L).\end{aligned}$$

Thus η is a valuation on $\mathcal{K}^n$. The continuity and invariance of μ_n, and the symmetry of $\mathbf{B}_n$ under Euclidean transformations show that η is a convex-continuous invariant valuation on $\mathcal{K}^n$, which according to the Extension Theorem 3.17 extends to a valuation in $\mathbf{Val}(n)$. By Hadwiger's Characterization Theorem we have that

$$\eta(K) = \sum_{i=0}^{n} c_i \widehat{\mu}_i(K),$$

for all $K \in \mathcal{K}^n$. Therefore, for $\epsilon > 0$ we have that

$$\mu_n(K + \epsilon\mathbf{B}_n) = \epsilon^n \mu_n\left(\frac{1}{\epsilon}K + \mathbf{B}_n\right) = \epsilon^n \sum_{i=0}^{n} c_i \widehat{\mu}_i(K) \frac{1}{\epsilon^i} = \sum_{i=0}^{n} c_i \widehat{\mu}_i(K) \epsilon^{n-i}.$$

In particular, if we let $K = C_n$ in the previous equation and comparing with the results of the previous Proposition, we find that

$$\sum_{i=0}^{n} c_i \mu_i(C_n) \epsilon^{n-i} = \sum_{i=0}^{n} \widehat{\mu}_i(C_n) \omega_{n-i} \epsilon^{n-i},$$

i.e. $c_i = \omega_{n-i}$. ☐

Theorem 5.4 (The intrinsic volumes of the unit ball)**.** *For* $0 \leq i \leq n$*,*

$$\widehat{\mu}_i(\mathbf{B}_n) = \binom{n}{i} \frac{\omega_n}{\omega_{n-i}} = \begin{bmatrix} n \\ i \end{bmatrix} \omega_i.$$

Proof. Applying the tube formula to to $K = \mathbf{B}_n$ we obtain

$$\begin{aligned}\sum_{i=0}^{n} \widehat{\mu}_i(\mathbf{B}_n) \omega_{n-i} \epsilon^{n-i} &= \mu_n(\mathbf{B}_n + \epsilon\mathbf{B}_n) = \mu_n\left((1+\epsilon)\mathbf{B}_n\right) \\ &= (1+\epsilon)^n \mu_n(\mathbf{B}_n) = \sum_{i=0}^{n} \binom{n}{i} \omega_n \epsilon^{n-i},\end{aligned}$$

for all $\epsilon > 0$. Comparing coefficients of powers of ϵ, we uncover the desired result. ☐

§5.2. **Universal Normalization and Crofton's Formulæ.** It turns out that the intrinsic volumes of the unit ball are the final piece of the puzzle in understanding the constants C_k^n in the formula $\widehat{\mu}_k^n = C_k^n \mu_k^n$, where

$$\mu_k^n = \mathcal{R}_{n,n-k}\mu_0 \text{ and } \widehat{\mu}_k^n = \mu_k \text{ on } \operatorname{Pix}(n).$$

We have the following very pleasant surprise.

Theorem 5.5. *For $0 \le k \le n$ and $K \in \mathcal{K}^n$,*

$$\widehat{\mu}_k = \widehat{\mu}_k^n = \mu_k^n$$

In other words, no more hats!

Proof. From Theorem 5.4 we deduce

$$C_k^n \mu_k^n(\mathbf{B}_n) = \widehat{\mu}_k^n = \omega_n \binom{n}{n-k} = \omega_n \binom{n}{k}.$$

On the other hand (3.8) shows that

$$\mu_k^n = \omega_k \binom{n}{k}.$$

Equating the two, we readily see that $C_k^n = 1$ for all $n, k \ge 0$. In fact, since both $\widehat{\mu}_k$ and μ_k are intrinsic, we have shown that $\widehat{\mu}_k = \mu_k$. ☐

We have thus proved that the intrinsic volumes $\widehat{\mu}_k$ on $\operatorname{Polycon}(n)$ are exactly the Radon Transforms $\mathcal{R}_{n,n-k}\mu_0$, a result to which we feel obliged to "tip our hats".

The following theorem is a generalization of the previous one.

Theorem 5.6 (Crofton's formula)**.** *For $0 \le i, j \le n$ and $K \in \operatorname{Polycon}(n)$,*

$$(\mathcal{R}_{n,n-i}\mu_j)(K) = \begin{bmatrix} i+j \\ j \end{bmatrix} \mu_{i+j}(K)$$

Proof. From the section before regarding Radon Transforms, we already know that $\mathcal{R}_{n,n-i}\mu_j \in \mathbf{Val}^{i+j}(n)$. By Corollary 4.9 we have that there exists a $c \in \mathbb{R}$ such that $\mathcal{R}_{n-i}^n \mu_j = c\mu_{i+j}$. To obtain this c we simply evaluate $(\mathcal{R}_{n,n-i}\mu_j)(\mathbf{B}_n)$.

$$\begin{aligned}
(\mathcal{R}_{n,n-i}\mu_j)(\mathbf{B}_n) &= \big(\mathcal{R}_{n,n-i}((\mathcal{R}_{n,n-i})\mu_0)\big)(\mathbf{B}_n) \\
&= \int_{\operatorname{Graff}(n,n-i)} \left(\int_{\operatorname{Graff}(V,n-i-j)} \mu_0(\mathbf{B}_n \cap W) d\lambda_{n-i-j}^{n-i}(W) \right) d\lambda_{n-i}^n(V) \\
&= \int_{\operatorname{Gr}(n,n-i)} \left(\int_{\operatorname{Gr}(V,n-i-j)} \int_{L^\perp} \mu_0(\mathbf{B}_n \cap (L+x)) dp \, d\nu_{n-i-j}^{n-i}(L) \right) d\nu_{n-i}^n(V) \\
&= \int_{\operatorname{Gr}(n,n-i)} \left(\int_{\operatorname{Gr}(V,n-i-j)} \int_{\mathbf{B}_{i+j}} dp \, d\nu_{n-i-j}^{n-i}(L) \right) d\nu_{n-i}^n(V) \\
&= \int_{\operatorname{Gr}(n,n-i)} \left(\begin{bmatrix} n-i-j \\ n-i \end{bmatrix} \omega_{i+j} \right) d\nu_{n-i}^n(V)
\end{aligned}$$

$$= \begin{bmatrix} n \\ n-i \end{bmatrix} \begin{bmatrix} n-i-j \\ n-i \end{bmatrix} \omega_{i+j}.$$

But we also have by Theorem 5.4 that

$$c\mu_{i+j}(\mathbf{B}_n) = c \begin{bmatrix} n \\ i+j \end{bmatrix} \omega_{i+j}$$

and thus

$$c = \begin{bmatrix} n \\ i+j \end{bmatrix}^{-1} \begin{bmatrix} n \\ n-i \end{bmatrix} \begin{bmatrix} n-i \\ n-i-j \end{bmatrix} = \begin{bmatrix} i+j \\ j \end{bmatrix}.$$

□

Remark 5.7. If we define a weighted Radon transform

$$\mathcal{R}^*_w : \mathbf{Val}(k) \to \mathbf{Val}(k+w), \quad \mathcal{R}^*_w := [w]!\mathcal{R}_{k+w,k}$$

and we set $\mu^*_i := [i]!\mu_i$ then we can rewrite Crofton's formulæ in the more symmetric form

$$\mu^*_{i+j} = \mathcal{R}^*_j \mu^*_i.$$

Note that μ^*_i are also intrinsic valuations, i.e. are independent of the dimension of the ambient space. □

§5.3. **The Mean Projection Formula Revisited.** The conclusion of Theorem 5.5 allows us to restate Cauchy's surface area formula as follows:

Theorem 5.8 (The mean projection formula)**.** *For* $0 \leq k \leq n$ *and* $C \in \mathcal{K}^n$*,*

$$(\mathcal{R}_{n,n-k}\mu_0)(C) = \mu_k(C) = \int_{\mathrm{Gr}(n,k)} \mu_k(C|V_0)\, d\nu^n_k(V_0).$$

□

It turns out that Cauchy's mean value formula is a special case of a more general formula.

Theorem 5.9 (Kubota)**.** *For* $0 \leq k \leq l \leq n$ *and* $C \in \mathcal{K}^n$*,*

$$\int_{\mathrm{Gr}(n,l)} \mu_k(C|V)\, d\nu^k_l(V) = \begin{bmatrix} n-k \\ l-k \end{bmatrix} \mu_k(C).$$

Proof. Define a valuation ν on $\mathcal{K}^n$ by

$$\nu(C) = \int_{\mathrm{Gr}(n,l)} \mu_k(C|V)\, d\nu^k_l(V).$$

Arguing as in the proof of Proposition 3.30 we deduce that $\nu \in \mathbf{Val}^k(n)$. By Corollary 4.9 there exists a constant $c \in \mathbb{R}$ such that $\nu = c\mu_k$. As before, we compute the constant c by considering what happens when $C = \mathbf{B}_n$.

$$c\mu_k(\mathbf{B}_n) = \nu(\mathbf{B}_n) = \int_{\mathrm{Gr}(n,l)} \mu_k(C|V)\, d\nu^k_l(V) = \mu_k(\mathbf{B}_l) \begin{bmatrix} n \\ l \end{bmatrix}.$$

Therefore

$$c = \frac{\mu_k(\mathbf{B}_l)}{\mu_k(\mathbf{B}_n)} \begin{bmatrix} n \\ l \end{bmatrix} = \begin{bmatrix} l \\ k \end{bmatrix} \omega_k \begin{bmatrix} n \\ k \end{bmatrix}^{-1} \frac{1}{\omega_k} \begin{bmatrix} n \\ l \end{bmatrix}$$

$$= \frac{[l]!}{[k]![l-k]!} \frac{[k]![n-k]!}{[n]!} \frac{[n]!}{[l]![n-l]!} = \frac{[n-k]!}{[n-l]![l-k]!} = \begin{bmatrix} n-k \\ l-k \end{bmatrix}.$$

□

§5.4. **Product Formulæ.** We now examine the intrinsic volumes on products.

Theorem 5.10. *Let* $0 \le k \le n$. *Let* $K \in \mathrm{Polycon}(k)$ *and* $L \in \mathrm{Polycon}(n-k)$. *Then*

$$\mu_i(K \times L) = \sum_{r+s=i} \mu_r(K)\mu_s(L).$$

Proof. The set function $\mu_i(K \times L)$ is a continuous valuation in each of the variables K and L when the other is fixed. Also, note that every rigid motion $\phi \in E_k$ is the restriction of some rigid motion $\Phi \in E_n$ that restricts to the identity on the orthogonal complement $\mathbb{R}^{n-k}$. Thus, by the invariance of μ_i,

$$\mu_i\big(\phi(K \times L)\big) = \mu_i\big(\Phi(K \times L)\big) = \mu_i(K \times L).$$

The characterization theorem tells us that for every L that there exist constants $c_r(L)$, depending on L, such that

$$\mu_i(K \times L) = \sum_{r=0}^{k} c_r(L)\mu_r(K),$$

for all $K \in \mathcal{K}^k$.

Repeating the same argument with fixed K and varying L, we see that the $c_r(L)$ are continuous invariant valuations on $\mathcal{K}^{n-k}$, so we apply the characterization theorem again, collect terms appropriately, and ultimately conclude that there are constants $c_{rs} \in \mathbb{R}$ such that

$$\mu_i(K \times L) = \sum_{r=0}^{k} \sum_{s=0}^{n-k} c_{rs}\mu_r(K)\mu_s(L),$$

for all $K \in \mathcal{K}^k$ and $L \in \mathcal{K}^{n-k}$.

Now we determine the constants using C_m, the unit m-dimensional cube. Let $\alpha, \beta \ge 0$. Then

$$\mu_i(\alpha C_k \times \beta C_{n-k}) = \sum_{r=0}^{k} \sum_{s=0}^{n-k} c_{rs}\mu_r(C_k)\mu_s(C_{n-k})\alpha^r\beta^s$$

$$= \sum_{r=0}^{k} \sum_{s=0}^{n-k} c_{rs} \binom{k}{r}\binom{n-k}{s} \alpha^r\beta^s.$$

On the other hand, we already know how to compute this from the product formula in Theorem 2.17

$$\mu_i(\alpha C_k \times \beta C_{n-k}) = \sum_{r+s=i} \binom{k}{r}\binom{n-k}{s} \alpha^r\beta^s.$$

Comparing these two, we see that for $0 \leq r \leq k$ and $0 \leq s \leq n-k$, we have $c_{rs} = 1$ if $r+s = i$ and $c_{rs} = 0$ otherwise.

Hence,

$$\mu_i(K \times L) = \sum_{r+s=i} \mu_r(K)\mu_s(L).$$

□

As a result of this theorem, we get the following corollary.

Corollary 5.11. *Suppose that* μ *is a convex-continuous invariant valuation on* $\mathrm{Polycon}(n)$ *such that*

$$\mu(K \times L) = \mu(K)\mu(L),$$

for all $K \in \mathrm{Polycon}(k)$, $L \in \mathrm{Polycon}(n-k)$, *where* $0 \leq k \leq n$. *Then either* $\mu = 0$ *or there exists* $c \in \mathbb{R}$ *such that*

$$\mu = \mu_0 + c\mu_1 + c^2\mu_2 + \cdots + c^n\mu_n.$$

Conversely, if μ *is a valuation satisfying*

$$\mu = \mu_0 + c\mu_1 + c^2\mu_2 + \cdots + c^n\mu_n,$$

then μ *satisfies*

$$\mu(K \times L) = \mu(K)\mu(L).$$

Proof. By the characterization theorem, there are real constants c_i such that

$$\mu = c_0\mu_0 + c_1\mu_1 + \cdots + c_n\mu_n.$$

Let C_k denote the unit cube in $\mathbb{R}^k$. Then for all $\alpha, \beta \geq 0$, the multiplication condition implies that

$$\mu(\alpha C_k \times \beta C_{n-k}) = \mu(\alpha C_k)\mu(\beta C_{n-k}) = \left(\sum_{r=0}^{k} c_r\alpha^r\mu_r(C_k)\right)\left(\sum_{s=0}^{n-k} c_s\beta^s\mu_s(C_{n-k})\right)$$

$$= \sum_{r=0}^{k}\sum_{s=0}^{n-k} c_r c_s \mu_r(C_k)\mu_s(C_{n-k})\alpha^r\beta^s.$$

On the other hand, by the previous theorem,

$$\mu(\alpha C_k \times \beta C_{n-k}) = \sum_{i=0}^{n} c_i\mu_i(\alpha C_k \times \beta C_{n-k}) = \sum_{i=0}^{n} c_i \sum_{r+s=i} \mu_r(C_k)\mu_s(C_{n-k})\alpha^r\beta^s$$

$$= \sum_{r=0}^{k}\sum_{s=0}^{n-k} c_{r+s}\mu_r(C_k)\mu_s(C_{n-k})\alpha^r\beta^s.$$

Comparing these polynomials in α, β, we see that the coefficients must be equal. Hence, $c_r c_s = c_{r+s}$, for all $0 \leq r, s \leq n$. Thus, $c_0 = c_{0+0} = c_0^2$. Hence, c_0 is either 0 or 1. If $c_0 = 0$, then $c_r = c_{r+0} = c_r c_0 = 0$, making μ zero. If $c_0 = 1$, then relabel $c_1 = c$. Then, for $r > 0$, $c_r = c_{1+\cdots+1} = c_1^r = c^r$, and we are done. The converse follows by simply applying the previous theorem. □

§5.5. **Computing Intrinsic Volumes.** Now that we have a better understanding of the intrinsic measures μ_i, it would be nice if we could compute the intrinsic volumes of various types of sets. We already know how to compute them for pixelations. We want to explain how to determine them for convex polytopes. In a later section we will explain how to compute them for arbitrary polytopes using triangulations and the Möbius inversion formula.

Suppose $P \in \mathcal{K}^n$ is a convex polytope. Using the tube formula (5.1) we deduce that for all $r > 0$

$$\mu_n(P + r\mathbf{B}) = \sum_{i=0}^{n} \mu_i(K)\omega_{n-i}r^{n-i},$$

where the superscript of $\mathbf{B}_n$ has been dropped for notational convenience. Suppose $x \in P + r\mathbf{B}$. Because P is compact and convex, there exists a unique point $x_P \in P$ such that

$$|x - x_P| \leq |x - y|, \ \ \forall y \in P.$$

If $x \in P$, then clearly $x = x_P$. If, on the other hand, $x \notin P$, then $x_P \in \partial P$. Furthermore, if $x \notin P$ and $y \in \partial P$, then $y = x_P \iff x - y \perp H$, where H is the support plane of P and $y \in P \cap H$. That is to say, the line connecting x and x_p must be perpendicular to the boundary.

Denote by $P_i(r)$ the set of all $x \in P + r\mathbf{B}$ such that x_P lies in the relative interior of an i-face of P. If x is in the relative interior of P then it is contained in an n-face of P, and consequently so is $x_P = x$. If x is on the boundary then it is in one of P's faces (and again, so is $x_P = x$). Finally, if x is in neither the interior of P nor the boundary, then as stated before there is a unique x_P on the boundary and hence in an i-face of P.

$$P + r\mathbf{B} = \bigcup_{i=0}^{n} P_i(r).$$

The uniqueness of x_P implies that this union is disjoint.

Denote by $F_i(P)$ the set of all i-faces of P. Let Q be a face of P, and let v be any outward unit normal to the boundary of P at y. Then we set

$$M(Q, r) = \{y + \delta v \mid y \in \text{relint}(Q),\ 0 \leq \delta \leq r\}.$$

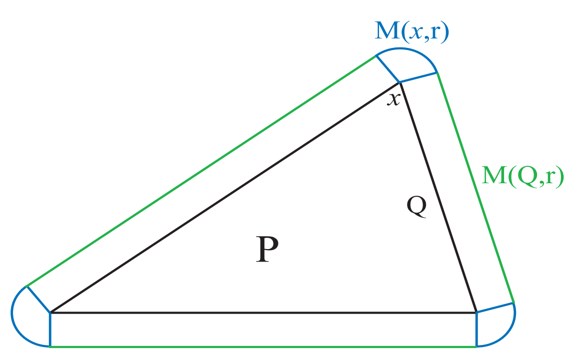

FIGURE 7. *A decomposition of* $P + r\mathbf{B}_2$.

Proposition 5.12.

$$P_i(r) = \bigcup_{Q \in F_i(P)} M(Q, r).$$

Proof. First, suppose $x \in P_i(r)$. From before, $x_p \in \text{relint}(Q)$ for some $Q \in F_i(P)$. If $x = x_P$, then $x \in \text{relint}(Q)$, and therefore $x \in M(Q, r)$ for any r. On the other hand, if $x \neq x_P$, then let v denote the unit vector parallel to the line between x and x_P. From the above discussion we know that v is perpendicular to the boundary of P at x_P, and therefore $x = x_P + \delta v$, where $\delta = |x - x_P|$. Thus we have that $P_i(r) \subset \bigcup_{Q \in F_i(P)} M(Q, r)$.

On the other hand, suppose $x \in M(Q, r)$. Then $x = y + \delta v$ for some $y \in \text{relint}(Q)$. If $\delta = 0$ then $x = y$ and is in $P_i(r)$. If $\delta \neq 0$, $x - y = \delta v$, and is therefore $\perp$ to the boundary of P at y. Therefore, $y = x_P$ and $x \in P_i(r)$. Therefore $P_i(r) \supset \bigcup_{Q \in F_i(P)} M(Q, r)$, and thus we have equality. □

For $A \subset \mathbb{R}^n$, let the *affine hull* of A be the intersection of all planes in $\mathbb{R}^n$ containing A. We then denote by $A^\perp$ the set of all vectors in $\mathbb{R}^n$ orthogonal to the affine hull of A.

Let $Q \in F_i(P)$. Then $Q^\perp$ has dimension $n - i$. Because of this, $M(Q, r)$ is like $M(Q, 1)$, except that it has been "stretched" by a factor of r in $n-i$ dimensions. Thus, $\mu_n(M(Q, r)) = r^{n-i}\mu_n(M(Q, 1))$. This fact, coupled with the fact that our $M(Q, r)$ are disjoint, gives us that

$$\mu_n(P_i(r)) = \mu_n\left(\bigcup_{Q \in F_i(P)} M(Q, r)\right) = r^{n-i}\mu_n\left(\bigcup_{Q \in F_i(P)} M(Q, 1)\right) = r^{n-i}\mu_n(P_i(1)).$$

Furthermore,

$$\mu_n(P + r\mathbf{B}) = \mu_n\left(\bigcup_{i=0}^{n} P_i(r)\right) = \sum_{i=0}^{n} \mu_n(P_i(r)) = \sum_{i=0}^{n} \mu_n(P_i(1))r^{n-i},$$

for all $r > 0$. Comparing with the tube formula (5.1), we see that

$$\sum_{i=0}^{n} \mu_i(K)\omega_{n-i}r^{n-i} = \sum_{i=0}^{n} \mu_n(P_i(1))r^{n-i}.$$

By comparing coefficients of powers or r, we get that

$$\mu_i(P) = \frac{\mu_n(P_i(1))}{\omega_{n-i}}. \tag{5.2}$$

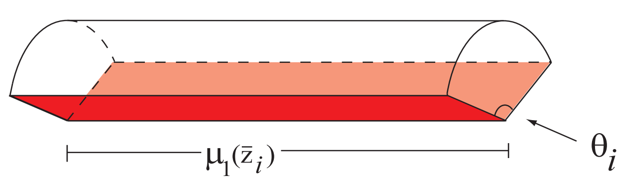

FIGURE 8. $M(\overline{z}_i, 1)$ *from Example 5.13.*

Example 5.13. Consider a general convex polyhedron P in $\mathbb{R}^3$ with edges $\overline{z}_1, \ldots, \overline{z}_m$. Each edge $\overline{z}_i$ is formed by two facets of P, denote these two faces Q_{i_1} and Q_{i_2}, and denote their outward unit normals by u_{i_1} and u_{i_2}, respectively. We then have that the volume of the section $M(\overline{z}_i, 1)$ is given by

$$\mu_3(M(\overline{z}_i, 1)) = \frac{\arccos(u_{i_1} \cdot u_{i_2})}{2\pi}(\pi(1)^2)(\mu_1(\overline{z}_i)) = \frac{\mu_1(\overline{z}_i)\theta_i}{2},$$

where $\theta_i = \arccos(u_{i_1} \cdot u_{i_2})$. Refer to Figure 8 for an example. Therefore

$$\mu_1(P) = \frac{\mu_3(P_1(1))}{\omega_2} = \frac{1}{2\pi}\sum_{i=1}^{m} \mu_1(\overline{z}_i)\theta_i.$$

This remarkable formula has immediate applications in specific polyhedra of interest.

For example, take an orthogonal parallelotope $P = a_1 \times a_2 \times a_3$. Clearly there are four sides of length a_i for each i, and the angle $\theta_i = \frac{\pi}{2}$ for each i. It is evident then that

$$\mu_1(P) = \frac{1}{2\pi}\sum_{i=1}^{m} 4a_i\frac{\pi}{2} = a_1 + a_2 + a_3.$$

6. **Simplicial complexes and polytopes**

At this point, we feel like we understand $\mathbf{Val}(n)$ to our satisfaction. We would now like to focus much more concretely on computing the these valuations for a special case of polyconvex sets, called polytopes. These are finite unions of convex polytopes. The inclusion-exclusion principle suggest dissecting these into smaller, and simpler pieces. Technically, this dissection operation is called triangulation, and the simpler pieces are called simplices. In this section, we formalize the notion of triangulation and investigate some of its combinatorial features.

§6.1. **Combinatorial Simplicial Complexes.**

A *combinatorial simplicial complex* (CSC for brevity) with vertex set in V is a collection K of non-empty subsets of the finite set V such that

$$\tau \in K \Longrightarrow \sigma \in K, \;\; \forall \sigma \subset \tau, \;\; \sigma \neq \emptyset.$$

The elements of K are called the *(open) faces* of the simplicial complex. If σ is a face then we define

$$\dim \sigma := \#\sigma - 1.$$

A subset σ of a face τ is also called a face of τ. If additionally

$$\#\sigma = \#\tau - 1$$

then we say that σ is an *(open) facet* of τ. If $\dim \sigma = 0$ then we say that σ is a vertex of K. We denote by V_K the collection of vertices of K. In other words, the vertices are exactly the singletons belonging to K. For simplicity we will write $v \in V_K$ instead of $\{v\} \in V_K$.

Two CSCs K and K' are called *isomorphic*, and we write this $K \cong K'$, if there exists a bijection from the set of vertices of K to the set of vertices of K' which induces a bijection between the set of faces of K and K'.

We denote by $\Sigma(V) \subset P(P(V))$ the collection of all CSC's with vertices in V. Observe that

$$K_1, K_2 \in \Sigma(V) \Longrightarrow K_1 \cap K_2, \;\; K_1 \cup K_2 \in \Sigma(V)$$

so that $\Sigma(V)$ is a sublattice of $P(P(V))$.

Example 6.1. For every subset $S \subset V$ define $\Delta_S \in \Sigma(V)$ to be the CSC consisting of all non-empty subsets of S. Δ_S is called the *elementary* simplex with vertex set S. Observe that

$$\Delta_{S_1} \cap \Delta_{S_2} = \Delta_{S_1 \cap S_2},$$

For every CSC $K \in \Sigma(V)$ the simplices Δ_σ, $\sigma \in K$ are called the *closed faces* of K.

We deduce that the collection

$$\Delta(V) := \Big\{ \Delta_S \mid S \subset V \Big\}$$

is a generating set of the lattice $\Sigma(V)$. □

Example 6.2. Let $K \in \Sigma(V)$. For every nonnegative integer m we define

$$K_m := \big\{ \sigma \in K \mid \dim \sigma \leq m \big\}.$$

K_m is a CSC called the *m-skeleton of K*. Observe that

$$K_0 \subset K_1 \subset \cdots, \quad K = \bigcup_{m \geq 0} K_m. \qquad \square$$

Definition 6.3. The *Euler characteristic* of a CSC $K \in \Sigma(V)$ is the integer

$$\chi(K) := \sum_{\sigma \in K} (-1)^{\dim \sigma}. \qquad \square$$

Proposition 6.4. *The map $\chi : \Sigma(K) \to \mathbb{Z}$, $K \mapsto \chi(K)$ is a valuation.*

Proof. For every $K \in \Sigma(V)$ and every nonnegative integer m we denote by $F_m(K)$ the collection of its m-dimensional faces, i.e.

$$F_m(K) := \{ \sigma \in K \mid \dim \sigma := m \}.$$

We then have

$$\chi(K) = \sum_{m \geq 0} (-1)^m \# F_m(K).$$

Then

$$F_m(K_1 \cup K_2) = F_m(K_1) \cup F_m(K_2), \quad F_m(K_1 \cap K_2) = F_m(K_1) \cap F_m(K_2)$$

and the claim of the proposition follows from the inclusion-exclusion property of the cardinality. $\square$

Example 6.5. Suppose Δ_S is a simplex. Then

$$\chi(\Delta_S) = \sum_{m \geq 0} \left(\sum_{\sigma \subset S,\, \#\sigma = m+1} (-1)^m \right) = \sum_{m \geq 0} (-1)^m \binom{n}{m+1} = 1. \qquad \square$$

§6.2. The Nerve of a Family of Sets.

Definition 6.6. Fix a finite set V. Consider a family

$$\mathcal{A} := \{A_v; \ \ v \in V\}$$

of subsets a set X parameterized by V.

(a) For every $\sigma \subset V$ we set

$$A_\sigma := \bigcap_{v \in \sigma} A_v.$$

The *nerve* of the family $\mathcal{A}$ is the collection

$$N(\mathcal{A}) := \{ \sigma \subset V \mid A_\sigma \neq \emptyset \}. \qquad \square$$

Clearly the nerve of a family $\mathcal{A} = \{ A_v \mid v \in V \}$ is a combinatorial simplicial complex with vertices in V.

Definition 6.7. Suppose V is a finite set and $K \in \Sigma(V)$. For every $\sigma \in K$ we define the *star* of σ in K to be the subset

$$\mathrm{St}_K(\sigma) := \{\tau \in K \mid \tau \supset \sigma\}.$$

Observe that

$$K = \bigcup_{v \in V_K} \mathrm{St}_K(v). \qquad \square$$

Proposition 6.8. *For every CSC $K \in \Sigma(V)$ we have the equalities*

$$\mathrm{St}_K(\sigma) = \bigcap_{v \in \sigma} \mathrm{St}_K(v), \;\; \forall \sigma \in K, \;\; K = \bigcup_{\sigma \in K} \Delta_\sigma.$$

In particular, we deduce that K is the nerve of the family of stars at vertices

$$\mathrm{St}_K := \{\mathrm{St}_K(v) \mid v \in V_K\}. \qquad \square$$

We can now rephrase of the inclusion-exclusion formula using the language of nerves.

Corollary 6.9. *Suppose $\mathcal{L}$ is a lattice of subsets of a set X, $\mu : \mathcal{L} \to R$ is a valuation into a commutative ring with 1. Then for every finite family $\mathcal{A} = (A_v)_{v \in V}$ of sets in $\mathcal{L}$ we have*

$$\mu\left(\bigcup_{v \in V} A_v\right) = \sum_{\sigma \in N(\mathcal{A})} (-1)^{\dim \sigma} \mu(A_\sigma).$$

In particular, for the universal valuation $A \mapsto I_A$ we have

$$I_{\cup_v A_v} = \sum_{\sigma \in N(\mathcal{A})} (-1)^{\dim \sigma} I_{A_\sigma}. \qquad \square$$

Corollary 6.10. *Suppose $\mathcal{C} = \{C_s\}_{s \in S}$ is a finite family of compact convex subsets of $\mathbb{R}^n$. Denote by $N(\mathcal{C})$ its nerve and set*

$$C := \bigcup_{s \in S} C_s.$$

Then

$$\mu_0(C) = \chi\big(N(\mathcal{C})\big).$$

In particular, if

$$\bigcap_{s \in S} C_s \neq \emptyset.$$

then

$$\mu_0(C) = 1,$$

where $\mu_0 : \mathrm{Polycon}(n) \to \mathbb{Z}$ denotes the Euler characteristic.

Proof. Let

$$C_\sigma = \bigcap_{s \in \sigma} C_s, \;\; \forall \sigma \subset S.$$

Note that if $\sigma \in N(\mathcal{C})$ then C_σ is nonempty, compact and convex and thus $\mu_0(C_\sigma) = 1$. Hence

$$\mu_0(C) = \sum (-1)^{\dim \sigma} \mu_0(C_\sigma) = \chi(\mathcal{C}).$$

If $C_\sigma \neq \emptyset \, \forall \sigma \subset S$ we deduce that $N(\mathcal{C})$ is the elementary simplex Δ_S. In particular

$$\mu_0(C) = \chi(\Delta_S) = 1.$$

□

§6.3. **Geometric Realizations of Combinatorial Simplicial Complexes.** Recall that a convex polytope in $\mathbb{R}^n$ is the convex hull of a finite collection of points, or equivalently, a compact set which is the intersection of finitely many half-spaces.

Definition 6.11. (a) A subset V

$$\{v_0, v_1, \dots, v_k\}$$

of $k+1$ points of a real vector space E is called *affinely independent* if the collection of vectors

$$\overrightarrow{v_0 v_1}, \dots, \overrightarrow{v_0 v_k}$$

is linearly independent.
(b) Let $0 \leq k \leq n$ be nonnegative integers. An *affine k-simplex* in $\mathbb{R}^n$ is the convex hull of a collection of $(k+1)$, affinely independent points $\{v_0, \cdots, v_k\}$ called the *vertices* of the simplex. We will denote it by $[v_0, v_1, \cdots, v_k]$.
(c) If S is an affine simplex in $\mathbb{R}^n$, then we denote by $V(S)$ the set of its vertices and we write $\dim \sigma := \#V(\sigma) - 1$. An affine simplex T is said to be a *face* of S, and we write this as $\tau \prec \sigma$ if $V(S) \subset V(T)$. □

Example 6.12. A set of three points is affinely independent if they are not collinear. A set of four points is affinely independent if they are not coplanar. If $v_0 \neq v_1$ then $[v_0, v_1]$ is the line segment connecting v_0 to v_1. If v_0, v_1, v_2 are not collinear then $[v_0, v_1, v_2]$ is the triangle spanned by v_0, v_1, v_2 (see Figure 9). □

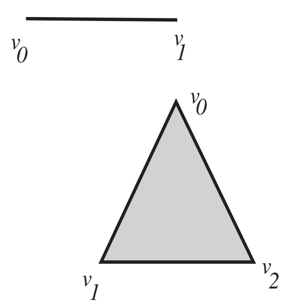

FIGURE 9. *A 1-simplex and a 2-simplex.*

Proposition 6.13. *Suppose $[v_0, \dots, v_k]$ is an affine k-simplex in $\mathbb{R}^n$. Then for every point $p \in [v_0, v_1, \dots, v_k]$ there exist real numbers $t_0, t_1, \cdots, t_k$* uniquely determined *by the requirements*

$$t_i \in [0,1], \quad \forall i = 0, 1, \cdots, n, \quad \sum_{i=0}^{k} t_i = 1, \quad p = \sum_{i=0}^{k} t_i v_i.$$

Proof. The existence follows from the fact that $[v_0, \dots, v_k]$ is the convex hull of the finite set $\{v_0, \dots, v_k\}$. To prove uniqueness, suppose

$$\sum_{i=0}^{k} s_i v_i = \sum_{i=0}^{k} t_i v_i.$$

Note that $v_0 = (s_0 + \cdots + s_k)v_0 = (t_0 + \cdots, t_k)v_0$ so that

$$\sum_{i=1}^{k} s_i(v_i - v_0) = \sum_{i=1}^{k} t_i(v_i - v_0), \quad v_i - v_0 = \overrightarrow{v_0 v_i}.$$

From the linear independence of the vectors $\overrightarrow{v_0 v_i}$ we deduce $s_i = t_i$, $\forall i = 1, \cdots, k$. Finally

$$s_0 = 1(s_1 + \cdots + s_k) = 1 - (t_1 + \cdots + t_k) = t_0.$$

□

The numbers (t_i) postulated by Proposition 6.13 are called the *barycentric coordinates* of the point p. We will denote them by $(t_i(p))$ In particular, the *barycenter* of a k simplex σ is the unique point b_σ whose barycentric coordinates are equal, i.e.

$$t_0(b_\sigma) = \cdots = t_k(b_\sigma) = \frac{1}{k+1}.$$

The *relative interior* of $\Delta[v_0, v_1, \dots, v_k]$ is the convex set

$$\Delta(v_0, v_1, \cdots, v_k) := \big\{ p \in \Delta[v_0, \cdots, v_k] \mid t_i(p) > 0, \ \ \forall i = 0, 1, \cdots, k \big\}.$$

Definition 6.14. An *affine simplicial complex* in $\mathbb{R}^n$ (ASC for brevity) is a pair $(C, \mathcal{T})$ satisfying the following conditions.

(a) C is a compact subset.

(b) $\mathcal{T}$ is a *triangulation of* C, i.e. a finite collection of affine simplices satisfying the conditions

(b1) If $T \in \mathcal{T}$ and S is a face of T then $S \in \mathcal{T}$.

(b2) If $T_0, T_1 \in \mathcal{T}$ then $T_0 \cap T_1$ is a face of both T_0 and T_1.

(b3) C is the union of all the affine simplices in $\mathcal{T}$. □

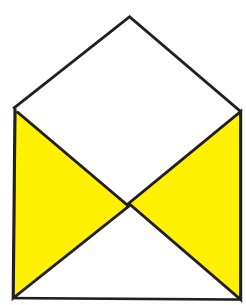

FIGURE 10. *An ASC in the plane.*

Remark 6.15. One can prove that any polytope in $\mathbb{R}^n$ admits a triangulation. □

In Figure 10 we depicted an ASC in the plane consisting of six simplices of dimension 0, nine simplices of dimension 1 and two simplices of dimension 2.

To an ASC $(C,\mathcal{T})$ we can associate in a natural way a CSC $K(C,\mathcal{T})$ as follows. The set of vertices V is the collection of 0-dimensional simplices in $\mathcal{T}$. Then

$$K=\big\{\,V(S)\mid S\in\mathcal{T}\,\big\}, \tag{6.1}$$

where we recall that $V(S)$ denotes the set of vertices of the affine simplex $S\in\mathcal{T}$.

Definition 6.16. Suppose K is a CSC with vertex set V. Then an *affine realization* of K is an ASC $(C,\mathcal{T})$ such that

$$K\cong K(C,\mathcal{T}).$$

□

Proposition 6.17. *Let V be a finite set. Then any CSC $K\in\Sigma(V)$ admits an affine realization.*

Proof. Denote by $\mathbb{R}^V$ the space of functions $f:V\to\mathbb{R}$. This is a vector space of the same dimension as V. It has a natural basis determined by the *Dirac functions* $\delta_u:V\to\mathbb{R}$, $u\in V$, where

$$\delta_u(v)=\begin{cases}1 & v=u\\ 0 & v\neq u\end{cases}.$$

The set $\{\delta_u\mid u\in V\}\subset\mathbb{R}^V$ is affinely independent.

For every $\sigma\in K$ we denote by $[\sigma]$ the affine simplex in $\mathbb{R}^V$ spanned by the set $\{\delta_u\mid u\in\sigma\}$. Now define

$$C=\bigcup_{\sigma\in K}[\sigma],\quad \mathcal{T}=\big\{[\sigma]\mid\sigma\in K\big\}.$$

Then $(C,\mathcal{T})$ is an ASC and by construction $K=K(C,\mathcal{T})$.

□

Suppose $(C,\mathcal{T})$ is an ASC and denote by K the associated CSC. Denote by $f_k=f_k(C,\mathcal{T})$. According to the Euler-Schläfli-Poincaré formula we have

$$\mu_0(C)=\sum_k(-1)^k f_k(C,\mathcal{T}).$$

The sum in the right hand side is precisely $\chi(K)$, the Euler characteristic of K. For this reason, we will use the symbols χ and μ_0 interchangeably to denote the Euler characteristic of a polytope.

7. The Möbius inversion formula

§7.1. **A Linear Algebra Interlude.** As in the previous section, for any finite set A we denote by $\mathbb{R}^A$ the space of functions $A \to \mathbb{R}$. This is a vector space and has a canonical basis given by the Dirac functions

$$\delta_a : A \to \mathbb{R},\ \ a \in A.$$

Note that to any linear transformation $T : \mathbb{R}^A \to \mathbb{R}^B$ we can associate a function

$$\mathcal{S} = \mathcal{S}_T : B \times A \to \mathbb{R}$$

uniquely determined by the equalities

$$T\delta_a = \sum_{b\in B} \mathcal{S}(b,a)\delta_b,\ \ a \in A. \tag{7.1}$$

We say that $\mathcal{S}_T$ is a *scattering matrix* of T. If $f : A \to \mathbb{R}$ is a function then

$$f = \sum_{a\in A} f(a)\delta_a$$

and we deduce that the function $Tf : B \to \mathbb{R}$ is given by the equalities

$$Tf = \sum_{a\in A,b'\in B} \mathcal{S}(b',a)f(a)\delta_{b'} \Longleftrightarrow (Tf)(b) = \sum_{a\in A} \mathcal{S}(b,a)f(a),\ \ \forall b \in B. \tag{7.2}$$

Conversely, to any map $\mathcal{S} : B \times A \to \mathbb{R}$ we can associate a linear transformation $T : \mathbb{R}^A \to \mathbb{R}^B$ whose action of δ_a is determined by (7.2).

Lemma 7.1. *Suppose A_0, A_1, A_2 are finite sets and*

$$T_0 : \mathbb{R}^{A_0} \to \mathbb{R}^{A_1},\ \ T_1 : \mathbb{R}^{A_1} \to \mathbb{R}^{A_2}$$

are linear transformations with scattering matrices

$$\mathcal{S}_0 : A_1 \times A_0 \to \mathbb{R},\ \ \mathcal{S}_1 : A_2 \times A_1 \to \mathbb{R}$$

Then the scattering matrix of $T_1 \circ T_0 : \mathbb{R}^{A_2} \times \mathbb{R}^{A_0} \to \mathbb{R}$ is the map $\mathcal{S}_1 * \mathcal{S}_0 : A_2 \times A_0 \to \mathbb{R}$ defined by

$$\mathcal{S}_1 * \mathcal{S}_0(a_2, a_0) = \sum_{a_1\in A_1} \mathcal{S}_1(a_2, a_1)\mathcal{S}_0(a_1, a_0).$$

Proof. Denote by $\mathcal{S}$ the scattering matrix of $T_1 \circ T_0$ so that

$$(T_1 \circ T_0)\delta_{a_0} = \sum_{a_2} \delta_{a_2}\mathcal{S}(a_2, a_0).$$

On the other hand

$$(T_1 \circ T_0)\delta_{a_0} = T_1\left(\sum_{a_1} \delta_{a_1}\mathcal{S}_0(a_1, a_0)\right)$$

$$= \sum_{a_2} \delta_{a_2}\left(\sum_{a_1} \mathcal{S}_1(a_2, a_1)\mathcal{S}_0(a_1, a_0)\right) = \sum_{a_2} \delta_{a_2}(\mathcal{S}_1 * \mathcal{S}_0)(a_2, a_0).$$

Comparing the above equalities we obtain the sought for identity

$$\mathcal{S}(a_2, a_0) = \big(\mathcal{S}_1 * \mathcal{S}_0\big)(a_2, a_0).$$

□

Lemma 7.2. *Suppose that A is a finite set equipped with a partial order $<$ (we use the convention that $a \not< a$, $\forall a \in A$). Suppose*

$$\mathcal{S} : A \times A \to \mathbb{R}$$

is strictly upper triangular*, i.e. it satisfies the condition*

$$\mathcal{S}(a_0, a_1) \neq 0 \Longrightarrow a_0 < a_1.$$

Then the linear transformation $T : \mathbb{R}^A \to \mathbb{R}^A$ determined by $\mathcal{S}$ is nilpotent, i.e. there exists a positive integer n such that

$$T^n = 0.$$

Proof. Let a, b be in A and denote by $\mathcal{S}_n$ the scattering matrix of T^n. By repeated application of the previous lemma,

$$\mathcal{S}_n(a,b) = \sum_{c_1,\dots,c_{n-1} \in A} \mathcal{S}(a, c_1)\mathcal{S}(c_1, c_2)\cdots\mathcal{S}(c_{n-1}, b).$$

Then since $\mathcal{S}$ is strictly upper triangular, we deduce that the above sums contains only 'monomials' of the form

$$\mathcal{S}(a, c_1)\mathcal{S}(c_1, c_2)\cdots\mathcal{S}(c_{n-1}, b), \quad c_i < c_{i+1}, \quad \forall i.$$

Thus, we really have that:

$$\mathcal{S}_n(a,b) = \sum_{a<c_1<c_2<\cdots<c_{n-1}<b \in A} \mathcal{S}(a, c_1)\mathcal{S}(c_1, c_2)\cdots\mathcal{S}(c_{n-1}, b).$$

So, since we are using that $a \not< a$, if we take $n > \#A$ we cannot find such a sequence, so $\mathcal{S}_n = 0$, meaning that $T^n = 0$. □

Lemma 7.3. *Suppose T is a finite dimensional vector space and $T : E \to E$ is a nilpotent linear transformation. Then the linear map $\mathbb{1}_E + T$ is invertible and*

$$(\mathbb{1}_E + T)^{-1} = \sum_{k \geq 0} (-1)^k T^k.$$

Proof. Since T is nilpotent, there is some positive integer m such that $T^m = 0$. We may even assume that m is *odd*. Thus

$$\mathbb{1}_E = \mathbb{1}_E + T^m = (\mathbb{1}_E + T)(\mathbb{1}_E - T + T^2 - \cdots + T^{m-1}) = (\mathbb{1}_E + T)\left(\sum_{k\geq 0} (-1)^k T^k\right).$$

□

§7.2. **The Möbius Function of a Simplicial Complex.** Suppose V is a finite set and $K \in \Sigma(V)$ is a CSC. The *zeta function* of K is the map

$$\zeta = \zeta_K : K \times K \to \mathbb{Z}, \;\; \zeta(\sigma, \tau) = \begin{cases} 1 & \sigma \preceq \tau \\ 0 & \sigma \not\preceq \tau \end{cases}.$$

Define $\xi = \xi_K : K \times K \to \mathbb{Z}$ by

$$\xi(\sigma, \tau) = \begin{cases} 1 & \sigma \precneqq \tau \\ 0 & \text{otherwise} \end{cases}.$$

ζ defines a linear map $Z : \mathbb{R}^K \to \mathbb{R}^K$ and ξ defines a linear map $\Xi : \mathbb{R}^K \to \mathbb{R}^K$. Note that for every function $f : K \to \mathbb{R}$ we have

$$(Zf)(\sigma) = \sum_{\tau \in K} \zeta(\sigma, \tau) f(\tau) = \sum_{\tau \succeq \sigma} f(\tau). \tag{7.3}$$

Proposition 7.4. *(a)* $Z = \mathbb{1} + \Xi$.
(b) The linear map Ξ *is nilpotent. In particular,* Z *is invertible.*
(c) Let $\mu : K \times K \to \mathbb{R}$ *the scattering matrix of* Z^{-1}*. Then* μ *satisfies the following conditions.*

$$\mu(\sigma, \tau) \neq 0 \Longrightarrow \sigma \preceq \tau, \tag{7.4a}$$

$$\mu(\sigma, \sigma) = 1, \;\; \forall \sigma \in K, \tag{7.4b}$$

$$\mu(\sigma, \tau) = -\sum_{\sigma \preceq \varphi \precneqq \tau} \mu(\sigma, \varphi). \tag{7.4c}$$

Proof. First, we denote by $\delta(\sigma, \tau)$ the delta function, that is,

$$\delta(\sigma, \tau) = \begin{cases} 1 & \sigma = \tau \\ 0 & \sigma \neq \tau \end{cases}$$

(a) From above we have

$$(Zf)(\sigma) = \sum_{\tau \in K} \zeta(\sigma, \tau) f(\tau) = \sum_{\tau \succeq \sigma} f(\tau) = f(\sigma) + \sum_{\tau \succneqq \sigma} f(\tau)$$

$$= \sum_{\tau \in K} \delta(\sigma, \tau) f(\tau) + \sum_{\tau \in K} \xi(\sigma, \tau) f(\tau) = (\Xi f)(\sigma) + (\mathbb{1} f)(\sigma) = \Big((\Xi + \mathbb{1}) f \Big)(\sigma).$$

(b) We have from before that Ξ is defined by a scattering matrix ξ which is strictly upper triangular, i.e.

$$\xi(\sigma, \tau) \neq 0 \Rightarrow \sigma \precneqq \tau.$$

By Lemma 7.2, we have that Ξ is nilpotent. Thus, because $Z = \mathbb{1} + \Xi$, by Lemma 7.3 we have that Z is invertible.
(c) From Lemma 7.3 we have that

$$Z^{-1} = (\mathbb{1} + \Xi)^{-1} = \sum_{k \geq 0} (-1)^k \Xi^k = \mathbb{1} - \Xi + \Xi^2 - \cdots$$

Let us examine Ξ^i. By an iteration of Lemma 7.1, we have that the scattering matrix for Ξ^i will be

$$\xi_i(\sigma,\tau) = \sum_{\varphi_1,\cdots,\varphi_{i-1}\in K} \xi(\sigma,\varphi_1)\xi(\varphi_1,\varphi_2)\cdots\xi(\varphi_{i-1},\tau)$$

This scattering matrix is, like ξ, strictly upper triangular. Note that this also implies that the $\xi_i(\sigma,\tau)$ gives a value of 0 for the diagonal terms $\sigma=\sigma$. Furthermore, the identity operator $\mathbb{1}$ has scattering matrix $\delta(\sigma,\tau)$, which gives a value of 1 on the diagonal terms and zero elsewhere. We therefore have proven equations (7.4a) and (7.4b). To prove the third, we convert the equality $\mathbb{1} = Z^{-1}Z$ into a statement about scattering matrices

$$\delta = \mu * \zeta.$$

By Lemma 7.1, we therefore have

$$\delta(\sigma,\tau) = \sum_{\varphi\in K} \mu(\sigma,\varphi)\zeta(\varphi,\tau)$$

Thus, for $\sigma \neq \tau$, we have that

$$0 = \sum_{\varphi\in K} \mu(\sigma,\varphi)\zeta(\varphi,\tau) = \sum_{\sigma\preceq\varphi\preceq\tau} \mu(\sigma,\varphi)$$

and thus

$$\mu(\sigma,\tau) = -\sum_{\sigma\preceq\varphi\precneqq\tau} \mu(\sigma,\varphi).$$

□

Definition 7.5. (a) Suppose A, B are two subsets of a set V. A *chain* of length $k \geq 0$ between A and B is a sequence of subsets

$$A = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_k = B.$$

We denote by $c_k(A,B)$ the number of chains of length k between A and B. We set

$$c_0(A,B) = \begin{cases} 0 & B \neq A \\ 1 & B = A \end{cases},$$

and

$$c(A,B) = \sum_k (-1)^k c_k(A,B).$$

(b) We set

$$c_k(n) := c_k(\emptyset, B), \quad c(n) = c(\emptyset, B),$$

where B is a set of cardinality n. □

Lemma 7.6. *If σ,τ are faces of the combinatorial simplicial complex K then*

$$\mu(\sigma,\tau) = c(\sigma,\tau).$$

Proof. We have that μ is the scattering matrix of

$$Z^{-1} = \sum_{k\geq 0}(-1)^k \Xi^k = \mathbb{1} - \Xi + \Xi^2 - \cdots ,$$

so μ should be equal to the scattering matrix of the right hand side, i.e.

$$\mu(\sigma,\tau) = \sum_{k\geq 0}(-1)^k \xi_k(\sigma,\tau),$$

where $\xi_k(\sigma,\tau)$ is, as before,

$$\xi_k(\sigma,\tau) = \sum_{\varphi_1,\cdots,\varphi_{k-1}\in K} \xi(\sigma,\varphi_1)\xi(\varphi_1,\varphi_2)\cdots\xi(\varphi_{k-1},\tau) = \sum_{\sigma=\varphi_0 \precneqq \varphi_1 \precneqq \cdots \precneqq \varphi_{k-1} \precneqq \varphi_k=\tau} 1 = c_k(\sigma,\tau).$$

Furthermore, because $c_0(\sigma,\tau) = \delta(\sigma,\tau)$, we have that

$$\mu(\sigma,\tau) = \sum_{k\geq 0}(-1)^k c_k(\sigma,\tau) = c(\sigma,\tau).$$

□

Lemma 7.7. *We have the following equalities.*
(a)

$$c_k(n) = \sum_{j>0}\binom{n}{j} c_{k-1}(n-j).$$

(b) $c(n) = (-1)^n$.
(c) If $A \subset B$ *are two finite sets then*

$$c(A,B) = (-1)^{\#B-\#A}.$$

Proof of a. We recall that $c_k(n)$ is the number of chains of length k in $C(B,\emptyset)$, where $|B| = n$. Then, $c_{k-1}(n-j)$ is the number of chains in $C(A,\emptyset)$, where $|A| = n-j$. Consider a chain of length $k-1$ in $C(A,\emptyset)$ where $A \subset B \subset V$: $\emptyset \subsetneq C_1 \subsetneq \cdots \subsetneq C_{k-1} = A$. There is only one option for turning it into a chain of length k in $C(B,\emptyset)$, namely: $\emptyset \subsetneq C_1 \subsetneq \cdots \subsetneq C_{k-1} \subsetneq C_k = B$. For each j, there are $\binom{n}{j}$ ways to choose the set J of elements to be taken out of B, i.e. to take $A = B \setminus J$, $|A| = n-j$. Then there are $c_{k-1}(n-j)$ chains of length k in $C(B,\emptyset)$ to be constructed as above. Thus,

$$c_k(n) = \sum_{j>0}\binom{n}{j} c_{k-1}(n-j)$$

□

Proof of b. We note that the statement is trivially true for $n = 0$, and we use this as a base case for induction. It is also important to note that for $n \neq 0$, $c_0(n) = 0$. By part a,

$$(-1)^k c_k(n) = (-1)^k \sum_{j>0}\binom{n}{j} c_{k-1}(n-j)$$

Then,

$$c(n) = \sum_{k\geq 0}(-1)^k c_k(n) = -\sum_{k\geq 0}\sum_{j>0}\binom{n}{j}(-1)^{k-1}c_{n-1}(n-j)$$

$$= -\sum_{j>0}\binom{n}{j}\sum_{k\geq 1}(-1)^{k-1}c_{k-1}(n-j)$$

$$= -\sum_{j>0}\binom{n}{j}\sum_{k\geq 0}(-1)^k c_k(n-j) = -\sum_{j>0}\binom{n}{j}c(n-j).$$

By the induction hypothesis,

$$-\sum_{j>0}\binom{n}{j}c(n-j) = -\sum_{j>0}\binom{n}{j}(-1)^{n-j} = (-1)^n - \sum_{j\geq 0}\binom{n}{j}(-1)^{n-j}.$$

The last sum is the Newton binomial expansion of $(1-1)^n$ so it is equal to zero. Hence

$$c(n) = (-1)^n.$$

□

Proof of c. Suppose $A = \{\alpha_1, \dots, \alpha_j\}$. If $A \subset B$, $B = \{\alpha_1, \dots, \alpha_j, \beta_1, \dots, \beta_m\}$. Then a chain in $c_k(A, B)$ can be identified with a chain of length k of the set

$$B \setminus A = \{\beta_1, \dots, \beta_m\}.$$

Therefore,

$$c_k(A, B) = c_k(\emptyset, B \setminus A) = c_k(\#B - \#A).$$

In particular, $c(A, B) = (-1)^{(\#B-\#A)}$. □

Theorem 7.8 (Möbius inversion formula)**.** *Let V be a finite set and $K \in \Sigma(V)$ suppose that $f, u : K \to \mathbb{R}$ are two functions satisfying the equality*

$$u(\sigma) = \sum_{\tau\succeq\sigma} f(\sigma), \quad \forall\sigma \in K.$$

Then

$$f(\sigma) = \sum_{\tau\succeq\sigma}(-1)^{\dim\tau-\dim\sigma}u(\tau).$$

Proof. Note that $u(\sigma) = (Zf)\sigma$ so that $Z^{-1}u = (Z^{-1}Z)f = f$. Using (7.2), and realizing that μ is the scattering matrix of Z^{-1} we deduce

$$f(\sigma) = (Z^{-1}u)(\sigma) = \sum_{\tau\in K}\mu(\sigma, \tau)u(\tau)$$

By Lemma 7.6,

$$f(\sigma) = \sum_{\tau\supseteq\sigma}c(\sigma, \tau)u(\tau)$$

Then, by Lemma 7.7,

$$f(\sigma) = \sum_{\tau\supseteq\sigma}(-1)^{\dim\tau-\dim\sigma}u(\tau).$$

□

Corollary 7.9. *Suppose* $(C, \mathcal{T})$ *is an ASC. Then*

$$I_C = \sum_{S \in \mathcal{T}} m(S) I_S$$

where

$$m(S) := \sum_{T \succeq S} (-1)^{\dim T - \dim S} = (-1)^{\dim S} \sum_{T \succeq S} (-1)^{\dim T}.$$

In particular, we deduce that the coefficients $m(S)$ *depend only on the combinatorial simplicial complex associated to* $(C, \mathcal{T})$.

Proof. Define $f, u : \mathcal{T} \to \mathbb{Z}$ by

$$f(T) := (-1)^{\dim T} \quad \text{and} \quad u(S) := (-1)^{\dim S} m(S) = \sum_{T \succeq S} f(T).$$

We apply the Möbius inversion formula and see that

$$f(S) = \sum_{T \succeq S} (-1)^{\dim T - \dim S} u(T) = (-1)^{\dim S} \sum_{T \succeq S} m(T).$$

Then,

$$1 = (-1)^{\dim S} f(S) = \sum_{T \succeq S} m(T). \tag{7.5}$$

Now consider the function $L : \mathbb{R}^n \to \mathbb{Z}$,

$$L(x) = \sum_{S \in \mathcal{T}} m(S) I_S(x).$$

Clearly, $L(x) = 0$ if $x \notin C$. Suppose $x \in C$. Denote by $\mathcal{T}_x$ the collection of simplices $T \in \mathcal{T}$ such that $x \in T$. Then

$$S_x := \bigcap_{T \in \mathcal{T}_x} T$$

is a simplex in $\mathcal{T}$ and in fact it is *the minimal* affine simplex in $\mathcal{T}$ containing x. We have

$$L(x) = \sum_{S \in \mathcal{T}} m(S) I_S(x) = \sum_{S \succeq S_x} m(S) I_S(x) = \sum_{S \succeq S_x} m(S) \overset{(7.5)}{=} 1.$$

Therefore,

$$\sum_{S \in \mathcal{T}} m(S) I_S = I_C$$

□

§7.3. **The Local Euler Characteristic.** We begin by defining an important concept in the theory of affine simplicial complexes, namely the concept of *barycentric subdivision*. As its definition is a bit involved we discuss first a simple example.

Example 7.10. (a) Suppose $[v_0, v_1]$ is an affine 1-simplex in some Euclidean space $\mathbb{R}^n$. Its barycenter is precisely the midpoint of the segment and we denote it by v_{01}. The barycentric subdivision of the line segment $[v_0 v_1]$ is the triangulation depicted at the top of Figure 11 consisting of the affine simplices.

$$v_0, v_{01}, v_1, [v_0, v_{01}], [v_{01}, v_1].$$

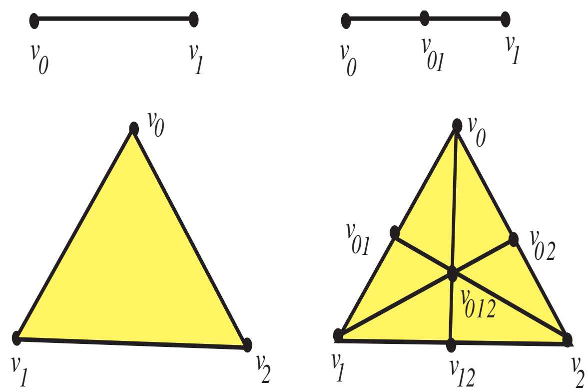

FIGURE 11. *Barycentric subdivisions*

(b) Suppose $[v_0, v_1, v_2]$ is an affine 2-simplex in some Euclidean space $\mathbb{R}^n$. We denote the barycenter of the face $[v_i, v_j]$ by $v_{ij} = v_{ji}$ and we denote by v_{012} the barycenter of the two simplex.

Observe that there exists a natural bijection between the faces of $[v_0, v_1, v_2]$ and the nonempty subsets of $\{0, 1, 2\}$. For every such subset $\sigma \subset \{0, 1, 2\}$ we denote by v_σ the barycenter of the face corresponding to σ. For example,

$$v_{\{0,1\}} = v_{01} \text{ etc..}$$

To any chain subsets of $\{0, 1, 2\}$, i.e. a strictly increasing family of subsets, we can associate a simplex. For example, to the increasing family

$$\{2\} \subset \{0, 2\} \subset \{0, 1, 2\}$$

we associate in Figure 11 the triangle $[v_2, v_{02}, v_{012}]$. We obtain in this fashion a triangulation of the 2-simplex $[v_0, v_1, v_2]$ whose simplices correspond bijectively to the chain of subsets.

The next two results follow directly from the definitions.

Lemma 7.11. *Suppose $[v_0, \cdots, v_k]$ is an affine simplex in $\mathbb{R}^n$. For every nonempty subset $\sigma \subset \{0, \cdots, k\}$ we denote by v_σ the barycenter of the affine simplex spanned by the points $\{v_s \mid s \in \sigma\}$. Then for every chain*

$$\emptyset \subsetneq \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_\ell$$

of subsets of $\{0, 1 \cdots, k\}$ the barycenters $v_{\sigma_0}, \cdots, v_{\sigma_\ell}$ are affinely independent.□

Proposition 7.12. *Suppose $(C, \mathcal{T})$ is an affine simplicial complex. For every chain*

$$S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k$$

of simplices in $\mathcal{T}$ we denote by $\Delta[S_0, \cdots, S_k]$ the affine simplex spanned by the barycenters of $S_0, \cdots, S_k$ and we denote $\mathcal{T}'$ the collection of all the simplices $\Delta[S_0, \cdots, S_k]$ obtained in this fashion. Then $\mathcal{T}'$ is a triangulation of $\mathcal{T}$.□

Definition 7.13. The triangulation $\mathcal{T}'$ constructed in Proposition 7.12 is called the *barycentric subdivision* of the triangulation $\mathcal{T}$. □

Definition 7.14. Suppose $(C, \mathcal{T})$ is an affine simplicial complex and v is a vertex of $\mathcal{T}$.
(a) The *star* of v in $\mathcal{T}$ is the collection of all simplices of $\mathcal{T}$ which contain v as a vertex. We denote the star by $\mathrm{St}_{\mathcal{T}}(v)$. For every simplex $S \in \mathrm{St}_{\mathcal{T}}(v)$ we denote by S/v the face of S opposite to v.
(b) The *link* of v in $\mathcal{T}$ is the polytope

$$\mathrm{lk}_{\mathcal{T}}(v) := \bigcup_{S \in \mathrm{St}(v)} S/v,$$

with the triangulation induced from $\mathcal{T}'$.
(c) For every face S of $\mathcal{T}$ we denote by b_S its barycenter. We define *link* of S in $\mathcal{T}$ to be the link of b_S in the barycentric subdivision

$$\mathrm{lk}_{\mathcal{T}}(S) := \mathrm{lk}_{\mathcal{T}'}(b_S).$$

(d) For every face S of $\mathcal{T}$ we define the *local Euler characteristic* of $(C, \mathcal{T})$ along S to be the integer

$$\chi_S(C, \mathcal{T}) := 1 - \chi\big(\,\mathrm{lk}_{\mathcal{T}}(S)\,\big).$$ □

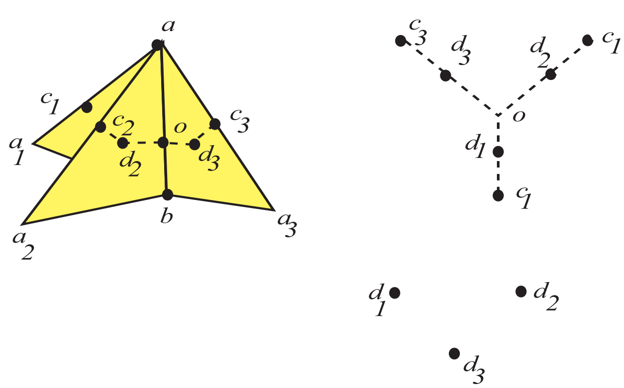

FIGURE 12. *An open book with three pages*

Example 7.15. Consider the polyconvex set C consisting of three triangles in space which have a common edge. Denote these triangles by $[a, b, a_i]$, $i = 1, 2, 3$ (see Figure 12).

We denote by o the barycenter of $[a, b]$, by c_i the barycenter of $[a, a_i]$ and by d_i the barycenter of $[a, b, a_i]$. Then the link of the vertex a is the star with tree arms joined at o depicted at the top right hand side. It has Euler characteristic 1. The link of $[a, b]$ is the set consisting of three points $\{d_1, d_2, d_3\}$. It has Euler characteristic 3 The local Euler characteristic along a is 0, while the local Euler characteristic along $[a, b]$ is -2. $\square$

Remark 7.16. Suppose $(C, \mathfrak{T})$ is an ASC. Denote by $V_{\mathfrak{T}} \subset \mathfrak{T}$ the set of vertices, i.e. the collection of 0-dimensional simplices. Recall that the associated combinatorial complex K consists of the collection

$$V(S), \quad S \in \mathfrak{T}$$

where $V(S)$ denotes the vertex set of S. The m-dimensional simplices the barycentric subdivision correspond bijectively to chains of length m in K.

Let S in $\mathfrak{T}$ and denote by σ its vertex set. The number of m-dimensional simplices in $\mathrm{St}_{\mathfrak{T}'}(b_S)$ is equal to

$$\sum_{\tau \supseteq \sigma} c_m(\sigma, \tau). \qquad \square$$

Proposition 7.17. *Suppose $(C, \mathfrak{T})$ is an ASC. Then for every $S \in \mathfrak{T}$ we have*

$$\chi_S(C, \mathfrak{T}) = \sum_{T \succeq S} (-1)^{\dim T - \dim S},$$

so that $I_C = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}) I_S$. In particular,

$$\sum_{S \in \mathfrak{T}} (-1)^{\dim S} = \chi(C) = \int I_C d\mu_0 = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}).$$

Proof. To begin, for a simplicial complex, K, let $F_m(K) = \{\text{faces of dimension } m \text{ in } K\}$ and recall that

$$\chi(K) = \sum_{m \geq 0} (-1)^m \# F_m(K).$$

The $(m-1)$-dimensional faces of $\mathrm{lk}_{\mathfrak{T}}(S)$ ($m \geq 1$) correspond bijectively to the m-dimensional simplices $T' \in \mathfrak{T}'$ containing b_S. According to Remark 7.16 there are $\sum_{T \succeq S} c_m(S, T)$ of them. Hence

$$F_{m-1}(\mathrm{lk}_{\mathfrak{T}}(S)) = \sum_{T \succeq S} c_m(S, T)$$

Hence

$$\chi(\mathrm{lk}_{\mathfrak{T}}(S)) = -\sum_{m > 0} (-1)^m \sum_{T \succeq S} c_m(S, T)$$

so that

$$\chi_S(C, \mathfrak{T}) = 1 - \chi(\mathrm{lk}_{\mathfrak{T}}(S)) = 1 + \sum_{m > 0} (-1)^m \sum_{T \succeq S} c_m(S, T)$$

$$= \sum_{m \geq 0} \sum_{T \succeq S} (-1)^m c_m(S, T) = \sum_{T \succeq S} \sum_{m \geq 0} (-1)^m c_m(S, T) = \sum_{T \succeq S} c(S, T) = \sum_{T \succeq S} (-1)^{\dim T - \dim S}.$$

$\square$

The above proof implies the following result.

Corollary 7.18. *For any vertex $v \in V_{\mathfrak{T}}$ we have*

$$\chi(\mathrm{lk}_{\mathfrak{T}}(v)) = -\sum_{S \supsetneq v} (-1)^{\dim S}. \qquad \square$$

Corollary 7.19. *Suppose $(C, \mathfrak{T})$ is an ASC in $\mathbb{R}^n$, R is an commutative ring with 1, and $\mu : \mathrm{Polycon}(n) \to R$ a valuation. Then*

$$\mu(C) = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}) \mu(S).$$

In particular, if we let μ be the Euler characteristic we deduce

$$\sum_{S \in \mathfrak{T}} (-1)^{\dim S} = \chi(C) = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}).$$

Proof. Denote by $\int d\mu$ the integral defined by μ. Then

$$\mu(C) = \int I_C d\mu = \int \left(\sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}) I_S \right) d\mu = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}) \int I_S d\mu = \sum_{S \in \mathfrak{T}} \chi_S(C, \mathfrak{T}) \mu(C).$$

$\square$

8. Morse theory on polytopes in $\mathbb{R}^3$

§8.1. **Linear Morse Functions on Polytopes.** Suppose $(C, \mathcal{T})$ is an ASC in $\mathbb{R}^3$. For simplicity we assume that all the faces of $\mathcal{T}$ have dimension < 3. Denote by $\langle -, - \rangle$ the canonical inner product in $\mathbb{R}^3$ and $\mathbf{S}^2$ the unit sphere centered at the origin of $\mathbb{R}^3$.

Any vector $u \in \mathbf{S}^2$ defines a linear map $L_u : \mathbb{R}^3 \to \mathbb{R}$ given by

$$L_u(x) := \langle u, x \rangle, \quad \forall x \in \mathbb{R}^3.$$

We denote by ℓ_u its restriction to C. We say the u is $\mathcal{T}$-*nondegenerate* if the restriction of L_u to the set of vertices of $\mathcal{T}$ is injective. Otherwise u is called $\mathcal{T}$-*degenerate*. We denote by $\Delta_{\mathcal{T}}$ the set of degenerate vectors.

Definition 8.1. A *linear Morse function* on $(C, \mathcal{T})$ is a function of the form ℓ_u, where$u$ is a $\mathcal{T}$-nondegenerate unit vector. □

Lemma 8.2 (Bertini-Sard)**.** *$\Delta_{\mathcal{T}}$ is a subset of $\mathbf{S}^2$ of zero (surface) area.*

Proof. Let $v_1, \cdots, v_k$ be the vertices of $\mathcal{T}$. For $u \in \mathbf{S}^2$ to be a $\mathcal{T}$-degenerate vector, it must be perpendicular to an edge connecting two of the v_i. The set of all such lines is a plane in $\mathbb{R}^3$, and the intersection of such a plane and $\mathbf{S}^2$ is a great circle. Therefore, $\Delta_{\mathcal{T}}$ is composed of at most $\binom{k}{2}$ great circles, meaning $\Delta_{\mathcal{T}}$ is a finite set and thus has zero surface area. □

Suppose u is $\mathcal{T}$-nondegenerate. For every $t \in \mathbb{R}$ we set

$$C_t = \left\{ x \in C \mid \ell_u(x) = t \right\}.$$

In other words, C_t is the intersection of C with the affine plane $H_{u,x_0} := \{x \mid \langle u, x \rangle = \langle u, x_0 \rangle = t\}$. We denote by $V_{\mathcal{T}}$ the set of vertices of $\mathcal{T}$. For every nondegenerate vector u the function ℓ_u defines an injection

$$\ell_u : V_{\mathcal{T}} \to \mathbb{R}.$$

Its image is a subset $K_u \subset \mathbb{R}$ called the *critical set* of ℓ_u. The elements of K_u are called the *critical values* of ℓ_u.

Lemma 8.3. *For every $t \in \mathbb{R}$ the slice C_t is a polytope of dimension ≤ 1.*

Proof. C_t is the intersection of C with the plane $H_t = \{\langle u, x \rangle\}$. H_t contains at most one vertex, so it must intersect each face transversally. The intersection with each face will have codimension 1 inside that face. Hence, no intersection can have dimension greater than 1. □

The above proof also shows that the slice C_t has a natural simplicial structure induced from the simplicial structure of C. The faces of C_t are the non-empty intersections of the faces of C with the hyperplane H_t.

Remark 8.4. Define a binary relation

$$R = R_{t,t+\epsilon} \subset V(C_{t+\epsilon}) \times V(C_t)$$

such that for $a \in V(C_{t+\epsilon})$ and $b \in V(C_t)$

$$a \ R_{t,t+\epsilon} \ b \Rightarrow a \text{ and } b \text{ lie on the same edge of } C$$

Observe that fixed t and for $\epsilon \ll 1$, the binary relation $R_{t,t+\epsilon}$ is the *graph of a map*

$$V(C_{t+\epsilon}) \to V(C_t)$$

which preserves the incidence relation. This means the following:

- $\forall a \in C_{t+\epsilon}$, $\exists$ a unique $b \in V(C_t)$ such that aRb, and
- If $[a_0, a_1, \cdots, a_k]$ is a face of $C_{t+\epsilon}$ and $a_i R b_i$ for $b_i \in V(C_t)$, then $[b_0, b_1, ..., b_k]$, is a face of C_t.

We will denote the map $V(C_{t+\epsilon}) \to V(C_t)$ by the same symbol R.

□

Denote by $\chi_u(t)$ the Euler characteristic of C_t. We know that

$$\chi(C) = \sum_t j_u(t),$$

where $j_u(t)$ denotes the *jump* of χ_u at t,

$$j_u(t) := \chi_u(t) - \chi_u(t+0).$$

Every nondegenerate vector u defines a map

$$j(u|-) : V_{\mathfrak{T}} \to \mathbb{Z}, \;\; j(u|x) := \text{the jump of } \chi_u \text{ at the critical value } \ell_u(x).$$

Lemma 8.5.

$$j_u(t) \neq 0 \Longrightarrow t \in K_u.$$

Proof. By definition $j_u(t) \neq 0 \Rightarrow \chi(C_t) \neq \chi(C_{t+0})$. We would like to better understand the relationship between C_t and $C_{t+\epsilon}$ for any $\epsilon \ll 1$. By Remark 8.4, for ϵ sufficiently small $R_{t,t+\epsilon}$ defines a map $V(C_{t+\epsilon}) \to V(C_t)$ which preserves the incidence relationship. Assume that $t \notin K_u$. All points in $V(C_t)$ and $V(C_{t+\epsilon})$ come from transversal intersections of edges of C. Because these edges cannot terminate anywhere inbetween those two points and because the intersection is transversal (and thus unique), for small ϵ we have that $V(C_{t+\epsilon}) \to V(C_t)$ is a bijection. Any edge which connects two vertices in $C_{t+\epsilon}$ is subsequently mapped by to the edge connecting the corresponding vertices in C_t. By the Euler-Schläfli-Poincaré formula, $\chi(C_{t+\epsilon}) = \chi(C_t)$. But this implies that $j_u(t) = 0$, a contradiction. Thus $t \in K_u$. □

Lemma 8.5 shows that

$$\chi(C) = \sum_{x \in V_{\mathfrak{T}}} j(u|x). \tag{8.1}$$

For every $x \in V_{\mathfrak{T}}$ we have a function

$$j(-|x) : \mathbf{S}^2 \setminus \Delta_{\mathfrak{T}} \to \mathbb{Z}, \;\; \mathbf{S}^2 \setminus \Delta_{\mathfrak{T}} \ni u \mapsto j(u|x).$$

Now, for every vertex x we set

$$\rho(x) := \frac{1}{4\pi} \int_{\mathbf{S}^2 \setminus \Delta_{\mathfrak{T}}} j(u|x) d\sigma(u), \tag{8.2}$$

where $d\sigma(u)$ denotes the area element on $\mathbf{S}^2$ such that

$$\text{Area}\,(\mathbf{S}^2) = \text{Area}\,(\mathbf{S}^2 \setminus \Delta_{\mathfrak{T}}) = 4\pi.$$

Integrating (8.1) over $\mathbf{S}^2 \setminus \Delta_{\mathcal{T}}$ we deduce

$$4\pi = \int_{\mathbf{S}^2\setminus\Delta_{\mathcal{T}}} \Big(\sum_{x\in V_{\mathcal{T}}} j(u|x)\Big) d\sigma(u) = \sum_{x\in V_{\mathcal{T}}} \int_{\mathbf{S}^2\setminus\Delta_{\mathcal{T}}} j(u|x) = 4\pi \sum_{x\in V_{\mathcal{T}}} \rho(x)$$

so that

$$\chi(C) = \sum_{x\in V_{\mathcal{T}}} \rho(x) \tag{8.3}$$

We dedicate the remainder of this section to giving a more concrete description of $\rho(x)$.

§8.2. **The Morse Index.** We begin by providing a combinatorial description of the jumps. Let $u \in \mathbf{S}^2$ be a $\mathcal{T}$-nondegenerate vector, x_0 a vertex of the triangulation $\mathcal{T}$. We set

$$t_0 := \ell_u(x_0) = \langle u, x_0\rangle.$$

We denote by $\mathrm{St}^+_{\mathcal{T}}(x_0, u)$ the collection of simplices in $\mathcal{T}$ which admit x_0 as a vertex and are contained in the half-space

$$H^+_{u,x_0} = \big\{ v \in \mathbb{R}^3 \mid \langle u, v\rangle \geq \langle u, x_0\rangle \big\}.$$

We define the *Morse index* of u at x_0 to be the integer

$$\mu(u|x_0) := \sum_{S\in \mathrm{St}^+_{\mathcal{T}}(x_0,u)} (-1)^{\dim S}.$$

Example 8.6. Consider the situation depicted in Figure 13.

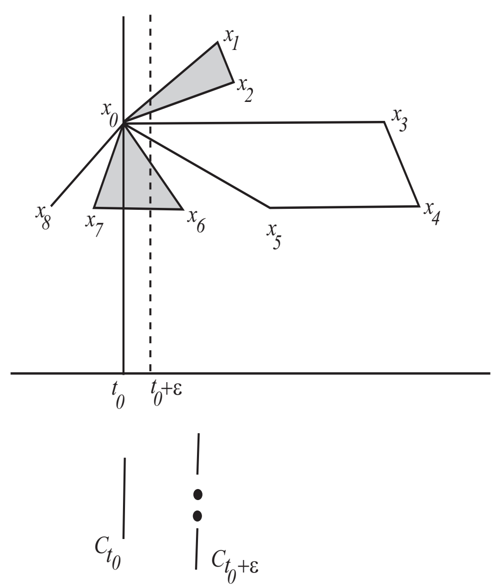

FIGURE 13. *Slicing a polytope.*

The hyperplanes $\ell_u = const$ are depicted as vertical lines. The slice C_{t_0} is a segment, and for every $\epsilon > 0$ sufficiently small the slice $C_{t_0+\epsilon}$ consists of two segments and two points. Hence

$$\chi_u(t_0) = 1, \;\; \chi_u(t_0+\epsilon) = 4, \;\; j(u|x_0) = j_u(t_0) = 1 - 4 = -3.$$

Now observe that $\mathrm{St}^+_{\mathcal{T}}(x_0, u)$ consists of the simplices

$$\{x_0\}, \quad [x_0, x_1], \quad [x_0, x_2], \quad [x_0, x_3], \quad [x_0, x_5], \quad [x_0, x_6], \quad [x_0, x_1, x_5].$$

We see that $\mathrm{St}^+_{\mathcal{T}}(x_0, u)$ consists of 1 simplex of dimension zero, 5 simplices of dimension 1 and one simplex of dimension 2 so that

$$\mu(u|x_0) = 1 - 5 + 1 = -3 = j(u|x_0).$$

We will see that the above equality is no accident. □

Proposition 8.7. *Let $(C, \mathcal{T})$ be a polytope in $\mathbb{R}^3$ such that all its simplices have dimension ≤ 2. Then for any $\mathcal{T}$-nondegenerate vector u and any vertex x_0 of $\mathcal{T}$ the jump of ℓ_u at x_0 is equal to the Morse index of u at x_0, i.e.*

$$j(u|x_0) = \mu(u|x_0).$$

Proof. By definition,

$$j(u|x_0) = \chi(C_{t_0}) - \chi(C_{t_0+0})$$

We will again utilize the binary relation R defined in Remark 8.4.

We have that t_0 is a critical value, so C_{t_0} contains a unique vertex x_0 of C. Denote by $R^{-1}(x_0)$ the set of all vertices of $V(C_{t_0+\epsilon})$ which are mapped to x_0 by $R_{t_0, t_0+\epsilon}$. Then the induced map $V(C_{t_0+\epsilon}) \setminus R^{-1}(x_0) \to V(C_{t_0}) \setminus \{x_0\}$ behaves as it did in Lemma 8.5 is a bijection on these sets and preserves the face incidence relation (see Figure 14). Using the Euler-Schläfli-Poincaré formula we then deduce that

$$\chi(C_{t_0}) - \chi(C_{t_0+\epsilon})$$
$$= \#\{x_0\} - \#\big(R^{-1}(x_0)\big) - \#\big\{\text{Edges in } C_{t+\epsilon} \text{ connecting vertices in } \ R^{-1}(x_0)\big\}.$$

Note here that

$$\#\big(R^{-1}(x_0)\big) = \#\big\{\text{Edges of } C \text{ inside } H^+_{u,x_0} \text{ and containing } x_0\big\},$$

and

$$\#\big\{\text{Edges in } C_{t+\epsilon} \text{ connecting vertices in } \ R^{-1}(x_0)\big\}$$
$$= \#\big\{ \text{ Triangles of } C \text{ inside } H^+_{u,x_0} \text{ and containig } x_0\big\}.$$

Thus we see that

$$j(u|x_0) = \#\{x_0\} - \#\big\{\text{Edges of } C \text{ inside } H^+_{u,x_0} \text{ and containing } x_0\big\}$$
$$+\#\big\{ \text{ Triangles of } C \text{ inside } H^+_{u,x_0} \text{ and containig } x_0\big\}$$
$$= \sum_{S \in \mathrm{St}^+_{\mathcal{T}}(x_0, u)} (-1)^{\dim S} = \mu(u|x_0).$$

□

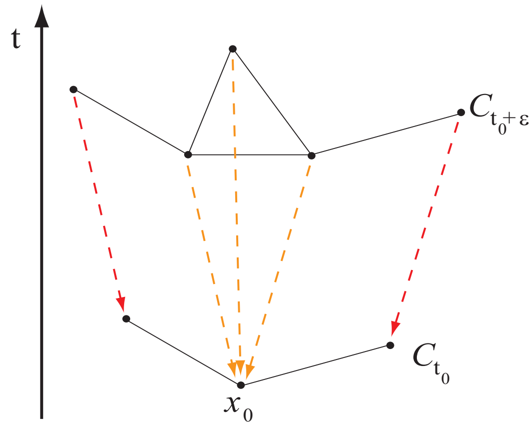

FIGURE 14. *The behavior of the map* $V(C_{t_0+\epsilon}) \to V(C_{t_0})$.

§8.3. **Combinatorial Curvature.** To formulate the notion of combinatorial curvature we need to introduce the notion of *conormal cone*. For $u, x \in \mathbb{R}^n$ define

$$H^+_{u,x} := \{ y \in \mathbb{R}^n \mid \langle u, y \rangle \geq \langle u, x \rangle \}.$$

Note that if $u \neq 0$ then $H^+_{u,x}$ is a half-space containing x on its boundary. u is normal to the boundary and points towards the interior of this half-space.

Definition 8.8. Suppose P is a convex polytope in $\mathbb{R}^n$ and x is a point in P. The *conormal cone* of $x \in C$ is the set

$$\mathcal{C}_x(P) = \mathcal{C}_x(P, \mathbb{R}^n) := \{ u \in \mathbb{R}^n \mid P \subset H^+_{u,x} \}. \qquad \square$$

We create an equivalent and more useful description of the conormal cone.

$$\mathcal{C}_x(P) := \{ u \in \mathbb{R}^n \mid P \subset H^+_{u,x} \} = \{ u \in \mathbb{R}^n \mid \langle u, y \rangle \geq \langle u, x \rangle, \ \ \forall y \in P \},$$

The following result follows immediately from the definition.

Proposition 8.9. *The conormal cone is a* convex cone, *i.e. it satisfies the conditions*

$$u \in \mathcal{C}_x(P), \ \ t \in [0, \infty) \Longrightarrow tu \in \mathcal{C}_x(P)$$

$$u_0, u_1 \in \mathcal{C}_x(P) \Longrightarrow u_0 + u_1 \in \mathcal{C}_x(P). \qquad \square$$

Proposition 8.9 shows that the conormal cone is an (infinite) union of rays (half-lines) starting at the origin. Each one of these rays intersects the unit sphere $\mathbf{S}^{n-1}$, and as a ray sweeps the cone P, its intersection with the sphere sweeps a region $\Omega_x(P)$ on the sphere,

$$\Omega_x(P) = \Omega_x(P, \mathbb{R}^n) := \mathcal{C}_x(P) \cap \mathbf{S}^{n-1}.$$

The more elaborate notation $\Omega_x(P, \mathbb{R}^n)$ is meant to emphasize that the region $\Omega_x(P)$ depends on the ambient space $\mathbb{R}^n$. Thus we also have

$$\Omega_x(P) = \{ u \in \mathbf{S}^{n-1} \mid \langle u, y \rangle \geq \langle u, x \rangle, \ \ \forall y \in P \}$$

We denote σ_{n-1} the total $(n-1)$-dimensional surface area of $\mathbf{S}^{n-1}$ and by $\omega_x(P)$ the $(n-1)$-dimensional "surface area" of $\Omega_x(P)$ divided by σ_{n-1},

$$\omega_x(P) := \frac{\operatorname{area}_{n-1}\big(\,\Omega_x(P)\,\big)}{\operatorname{area}_{n-1}(\mathbf{S}^{n-1})} = \frac{\operatorname{area}_{n-1}\big(\,\Omega_x(P)\,\big)}{\sigma_{n-1}}.$$

Remark 8.10. One can show that $\omega_x(P)$ is independent of the dimension n of the ambient space $\mathbb{R}^n$, i.e. if we regard P as a polytope in an Euclidean space $\mathbb{R}^N \supset \mathbb{R}^n$ then we obtain the same result for $\omega_x(P)$. We will not pursue this aspect here. □

Proposition 8.11. *(a) If $P \subset \mathbb{R}^3$ is a zero simplex $[x]$ then*

$$\omega_x(x) = 1.$$

(b) If $P = [x_0, x_1] \subset \mathbb{R}^3$ is a 1-simplex then

$$\omega_{x_0}([x_0, x_1]) = \frac{1}{2}.$$

(c) If $P = [x_0, x_1, x_2] \subset \mathbb{R}^3$ is a 2-simplex and the angle at the vertex x_i is $r_i\pi$, $r_i \in (0,1)$ then

$$r_0 + r_1 + r_2 = 1, \ \ \omega_{x_i}([x_0, x_1, x_2]) = \frac{1}{2} - \frac{r_i}{2}.$$

Proof. (a) For $[x] \subset \mathbb{R}^3$ a singleton, clearly

$$\Omega_{[x]}(P) = \big\{ u \in \mathbf{S}^{n-1} \mid \langle u, y\rangle \geq \langle u, x\rangle, \ \ \forall y \in [x] \big\} = \big\{ u \in \mathbb{S}^{n-1} \mid \langle u, x\rangle \geq \langle u, x\rangle \,\big\} = \mathbf{S}^{n-1}$$

and thus

$$\omega_{[x]}(P) = 1$$

(b) We can fix coordinates such that $[x_0, x_1]$ lies horizontal with x_0 located at the origin. It is then easy to see that the conormal cone is the half-space with boundary perpendicular to $[x_0, x_1]$ passing through x_0 with rays pointing "towards" x_1. Therefore, $\omega_{x_0} = \frac{1}{2}$.

(c) Without loss of generality, we can consider ω_{x_0}. We fix coordinates such that x_0 lies at the origin and $[x_0, x_1, x_2]$ lies in a plane with $[x_0, x_1]$ lying along an axis. The conormal cones of $[x_0, x_1]$ and $[x_0, x_2]$ are each a half-space as described above. The conormal cone of $[x_0, x_1, x_2]$ is then the intersection of these two cones. This intersection is bounded by planes determined by the perpendiculars to $[x_0, x_1]$ and $[x_0, x_1]$, both passing through x_0. In other words, the intersection of the conormal cone with the sphere is a lune. Call the angle of the opening of the lune θ. The angles between the perpendicular lines and the sides of the triangle are given by $\frac{\pi}{2} - \pi r_0$ (possibly negative), and the total angle is

$$\theta = \left(\frac{\pi}{2} - \pi r_0\right) + \left(\frac{\pi}{2} - \pi r_0\right) + \pi r_0 = \pi - \pi r_0$$

The area of the lune defined by θ_0 is, in spherical coordinates,

$$\int_0^{\theta_0} \left(\int_0^{\pi} \sin(\phi) d\phi \right) d\theta = 2\theta_0$$

Thus

$$\omega_{x_0} = \frac{2\theta}{4\pi} = \frac{1}{2} - \frac{r_1}{2}.$$

□

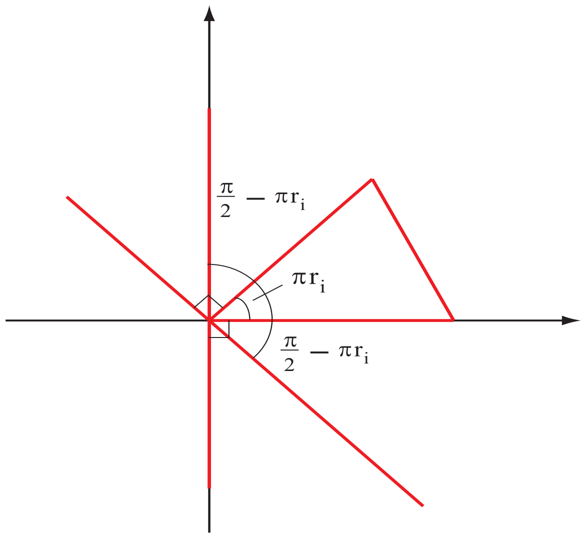

FIGURE 15. *The angle between the perpendiculars of* $[x_0, x_1]$ *and* $[x_0, x_2]$.

Suppose $(C, \mathcal{T})$ is an affine simplicial complex in $\mathbb{R}^3$ such that all the simplices in $\mathcal{T}$ have dimension ≤ 2. Recall that for every vertex $v \in V_{\mathcal{T}}$ we defined its star to be the collection of all simplices in $\mathcal{T}$ which admit v as a vertex. We now define the *combinatorial curvature* of $(C, \mathcal{T})$ at $v \in V_{\mathcal{T}}$ to be the quantity

$$\kappa(v) := \sum_{S \in \mathrm{St}_{\mathcal{T}}(v)} (-1)^{\dim S} \omega_v(S).$$

Example 8.12. (a) Consider a rectangle $A_1A_2A_3A_4$. Then any interior point O determines a triangulation of the rectangle as in Figure 16.

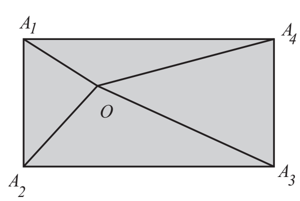

FIGURE 16. *A simple triangulation of a rectangle.*

Suppose that $\measuredangle(A_iOA_{i+1}) = r_i\pi$, $i = 1, 2, 3, 4$ so that

$$r_1 + r_2 + r_3 + r_4 = 2.$$

The star of O in the above triangulation consists of the simplices

$$\{O\}, \quad [OA_i], \quad [OA_iA_{i+1}], \quad i = 1, 2, 3, 4.$$

Then

$$\omega_O(O) = 1, \ \ \omega_O([OA_i]) = \frac{1}{2}, \ \ \omega_O([OA_iA_{i+1}]) = \frac{1}{2} - \frac{r_i}{2}.$$

We deduce that the combinatorial curvature at O is

$$1 - \sum_{i=1}^{4} \frac{1}{2} + \sum_{i=1}^{4} \Big(\frac{1}{2} - \frac{r_i}{2} \Big) = 1 - 2 + 2 - \frac{1}{2}(r_1 + r_2 + r_3 + r_4) = 0.$$

This corresponds to our intuition that a rectangle is "flat". Note that the above equality can be written as

$$2\pi\kappa(O) = 2\pi - \sum_{i=1}^{4} \measuredangle(A_iOA_{i+1}).$$

(b) Suppose $(C, \mathcal{T})$ is the standard triangulation of the boundary of a tetrahedron $[v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ (see Figure 17).

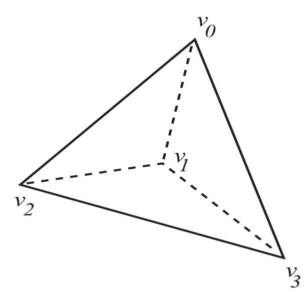

FIGURE 17. *The boundary of a tetrahedron.*

Set

$$\theta_{ijk} = \measuredangle(v_iv_jv_k), \ \ i, j, k = 0, 1, 2, 3..$$

Then the star of v_0 consists of the simplices

$$\{v_0\}, \ \ [v_0v_i], \ \ [v_0v_iv_j], \ \ i, j \in \{1, 2, 3\}, \ \ i \neq j.$$

We deduce

$$\omega_{v_0}(v_0) = 1, \ \ \omega_{v_0}([v_0v_i]) = \frac{1}{2}, \ \ \omega_{v_0}([v_0v_iv_j]) = \frac{1}{2\pi}(\pi - \theta_{i0j})$$

and

$$\kappa(v_0) = 1 - \frac{3}{2} + \frac{1}{2\pi} \sum_{1 \leq i < j \leq 3} (\pi - \theta_{i0j}) = \frac{1}{2\pi} \Bigg(2\pi - \sum_{1 \leq i < j \leq 3} \theta_{i0j} \Bigg).$$

Hence

$$2\pi\kappa(v_0) = 2\pi - \text{the sum of the angles at } v_0.$$

This resembles the formula we found in (a) and suggests an interpretation of the curvature as a measure of deviation from flatness. Note that in the limiting case when the vertex v_0 converges to a point in the interior of $[v_1v_2v_3]$ the sum of the angles at v_0 converges to 2π, and the boundary of the tetrahedron is "less and less curved" at v_0. On the other hand, if the tetrahedron is "very sharp" at v_0, the sum of the angles at v_0 is very small and the curvature approaches 2π. □

Motivated by the above example we introduce the notion of *angular defect*.

Definition 8.13. Suppose $(C, \mathcal{T})$ is an affine simplicial complex in $\mathbb{R}^3$ such that all simplices have dimension ≤ 2. For every vertex v of $\mathcal{T}$. we denote by $\Theta(v)$ the sum of the angles at v of the triangles in $\mathcal{T}$ which have v as a vertex. The *defect* at v is the quantity

$$\mathbf{def}(v) := 2\pi - \Theta(v).$$

□

Proposition 8.14. *If $(C, \mathcal{T})$ is an affine simplicial complex in $\mathbb{R}^3$ such that all simplices have dimension ≤ 2 then for every vertex v of $\mathcal{T}$ we have*

$$\kappa(v) = \frac{1}{2\pi}\mathbf{def}(v) - \frac{1}{2}\chi(\,\mathrm{lk}_{\mathcal{T}}(v)\,).$$

Proof. Using Proposition 8.11 we deduce that

$$\begin{aligned}\kappa(v) &= 1 - \frac{1}{2}\#\{\text{edges at } v\} + \frac{1}{2}\#\{\text{triangles at } v\} - \frac{1}{2\pi}\Theta(v)\\ &= \frac{1}{2\pi}\mathbf{def}(v) - \frac{1}{2}\Big(\#\{\text{edges at } v\} - \#\{\text{triangles at } v\}\Big)\end{aligned}$$

The proposition now follows from Corollary 7.18 which states that

$$\chi(\,\mathrm{lk}_{\mathcal{T}}(v)\,) = \Big(\#\{\text{edges at } v\} - \#\{\text{triangles at } v\}\Big).$$

□

Theorem 8.15 (Combinatorial Gauss-Bonnet)**.** *If $(C, \mathcal{T})$ is an affine simplicial complex in $\mathbb{R}^3$ such that all the simplices in $\mathcal{T}$ have dimension ≤ 2 then*

$$\chi(C) = \sum_{v \in V_{\mathcal{T}}} \kappa(v) = \frac{1}{2\pi}\sum_{v \in V_{\mathcal{T}}}\mathbf{def}(v) - \frac{1}{2}\sum_{v \in V_{\mathcal{T}}}\chi(\,\mathrm{lk}_{\mathcal{T}}(v)\,).$$

Proof. Let V represent the number of vertices in C, E the number of edges in C and T the number of triangles in C. Then we denote by E_v, T_v the number of edges and triangles in $\mathrm{St}_{\mathcal{T}}(v)$ for $v \in V_{\mathcal{T}}$, respectively. We note that

$$\chi(\mathrm{lk}_{\mathcal{T}}(v)) = \sum_{S \in \mathrm{St}_{\mathcal{T}}(v) \setminus v} (-1)^{\dim S - 1} = E_v - T_v$$

We note that each edge is in two stars and each triangle in three. Then,

$$-\frac{1}{2}\sum_{v \in V_{\mathcal{T}}}\chi(\mathrm{lk}_{\mathcal{T}}(v)) = -E + \frac{3}{2}T$$

Recall that $\mathbf{def}(v) = 2\pi - \Theta(v)$. Therefore, $\frac{1}{2\pi}\mathbf{def}(v) = 1 - \frac{\Theta(v)}{2\pi}$. Thus,

$$\sum_{v \in V_{\mathcal{T}}}\frac{1}{2\pi}\mathbf{def}(v) = \sum_{v \in V_{\mathcal{T}}} 1 - \frac{\Theta(v)}{2\pi} = V - \frac{T}{2}$$

Consequently,

$$\sum_{v \in V_{\mathcal{T}}} \kappa(v) = V - E + T = \chi(C)$$

□

Recall that a combinatorial surface is an ASC with simplices of dimension ≤ 2 such that every edge is the face of exactly two triangles. In this case, the link of every vertex is a simple cycle, that is a 1-dimensional ASC whose vertex set has a cyclical ordering

$$v_1, v_2, \cdots, v_n, v_{n+1} = v_1$$

and whose only edges are $[v_i, v_{i+1}]$, $i = 1, \cdots, n$. The Euler characteristic of such a cycle is zero. We thus obtain the following result of T. Banchoff, [**Ban**]

Corollary 8.16. *If C is a combinatorial surface with vertex set V then*

$$\chi(C) = \frac{1}{2\pi} \sum_{v \in V} \mathbf{def}(v).$$

□

We can now finally close the circle and relate the combinatorial curvature to the average Morse index.

Theorem 8.17 (Microlocal Gauss-Bonnet)**.** *If $(C, \mathcal{T})$ is an affine simplicial complex in $\mathbb{R}^3$ such that all the simplices in $\mathcal{T}$ have dimension ≤ 2 then*

$$\kappa(v) = \rho(v), \quad \forall v \in V_{\mathcal{T}},$$

where ρ is defined by (8.2).

Proof. Fix $x \in V_{\mathcal{T}}$. We now recall some previous ideas. Proposition 8.7 tells us that for any $u \in \mathbf{S}^2 \setminus \Delta_{\mathcal{T}}$,

$$\sum_{S \in \mathrm{St}_{\mathcal{T}}(x,u)} (-1)^{\dim S} = \mu(u|x) = j(u|x).$$

We then recall from the proof of Lemma 8.2 that $\Delta_{\mathcal{T}}$ is a finite union of great circles on $\mathbf{S}^2$. Thus, $\mathbf{S}^2 \setminus \Delta_{\mathcal{T}}$ consists of a finite union of chambers, $A_1, \ldots, A_m$. We now note that for any i, $1 \leq i \leq m$, $\mathrm{St}^+_{\mathcal{T}}(x,u) = \mathrm{St}^+_{\mathcal{T}}(x,v)$ for any $u, v \in A_i$. So, letting v_i be an element in A_i, we have:

$$\rho(x) = \frac{1}{4\pi} \int_{\mathbf{S}^2 \setminus \Delta_{\mathcal{T}}} j(u|x) d\sigma(u) = \frac{1}{4\pi} \int_{\mathbf{S}^2 \setminus \Delta_{\mathcal{T}}} \mu(u|x) d\sigma(u)$$

$$= \frac{1}{4\pi} \int_{\mathbf{S}^2 \setminus \Delta_{\mathcal{T}}} \sum_{S \in \mathrm{St}^+_{\mathcal{T}}(x,u)} (-1)^{\dim S} d\sigma(u) = \frac{1}{4\pi} \sum_{i=1}^{m} \int_{A_i} \sum_{S \in \mathrm{St}^+_{\mathcal{T}}(x,u)} (-1)^{\dim S} d\sigma(u)$$

$$= \frac{1}{4\pi} \sum_{i=1}^{m} \sum_{S \in \mathrm{St}^+_{\mathcal{T}}(x,v_i)} (-1)^{\dim S} \int_{A_i} d\sigma(u).$$

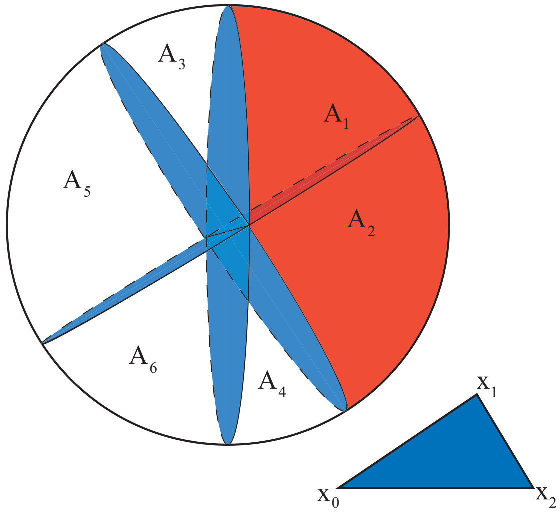

FIGURE 18. *The chambers which coincide with the conormal section* $\mathcal{C}_x(S) \cap \mathbf{S}^2$.

Considering this sum, we see that every term of it has a $(-1)^{\dim S}$ in it for some $S \in \mathrm{St}_{\mathcal{T}}(x)$. We also notice that every $S \in \mathrm{St}_{\mathcal{T}}(x)$ appears at least once. So, we expand the sum, collect the coefficients for each $(-1)^{\dim S}$ and if we set

$$k_S := \#\{i; \quad S \in \mathrm{St}^+(x, v_i)\}$$

we obtain

$$\rho(x) = \frac{1}{4\pi}\sum_{i=1}^{m} \sum_{S \in \mathrm{St}^+_{\mathcal{T}}(x,v_i)} (-1)^{\dim S} \int_{A_i} d\sigma(u) = \frac{1}{4\pi} \sum_{S \in \mathrm{St}_{\mathcal{T}}(x)} (-1)^{\dim S} \left(\sum_{j=1}^{k_S} \int_{A_{i_j}} d\sigma(u) \right).$$

Now we consider the sum $\sum_{j=1}^{k_S} \int_{A_{i_j}} d\sigma(u)$. This is the area of the set of vectors, u, on the unit sphere such that $S \in \mathrm{St}^+_{\mathcal{T}}(x, u)$. That is, this sum is the area of the set of vectors, u, on the unit sphere such that S lies in $H^+_{u,x}$. But this is precisely the area of $\mathcal{C}_x(S) \cap \mathbf{S}^2$ (see Figure 18). Now recall that $\omega_x(S) = \frac{\text{area}(\mathcal{C}_x(S)\cap\mathbf{S}^2)}{4\pi}$ (since the area of the unit sphere in $\mathbb{R}^3$ is 4π). Thus, we have:

$$\rho(x) = \frac{1}{4\pi} \sum_{S \in \mathrm{St}_{\mathcal{T}}(x)} (-1)^{\dim S} \left(\sum_{j=1}^{k_S} \int_{A_{i_j}} d\sigma(u) \right) = \frac{1}{4\pi} \sum_{S \in \mathrm{St}_{\mathcal{T}}(x)} (-1)^{\dim S} (\text{area}(\mathcal{C}_x(S) \cap \mathbf{S}^2))$$

$$= \sum_{S \in \mathrm{St}_{\mathcal{T}}(x)} (-1)^{\dim S} \omega_x(S) = \kappa(x).$$

□

Remark 8.18. The above equality can be interpreted as saying that the curvature at a vertex x_0 is the average Morse index at x_0 of a linear Morse function ℓ_u. □

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