I. Multiple Choice Questions

1. (a) The maximum velocity of the bullet would be obtained if we assume that the second disk only undergoes a quarter-turn while the bullet is in flight between the disks. The time it takes for the bullet to fly between the disks is given by $\frac{1}{4}$ revolution/(200 rev/s)= 0.00125 sec. In this time, the bullet travels 0.5 m, so $v = d/t = 400$ m/s.

2. (a) The force is zero when $\frac{dU}{dx} = 0$, since $F = -\frac{dU}{dx}$. So, all of the points b, d, f, and h have zero force at their positions.

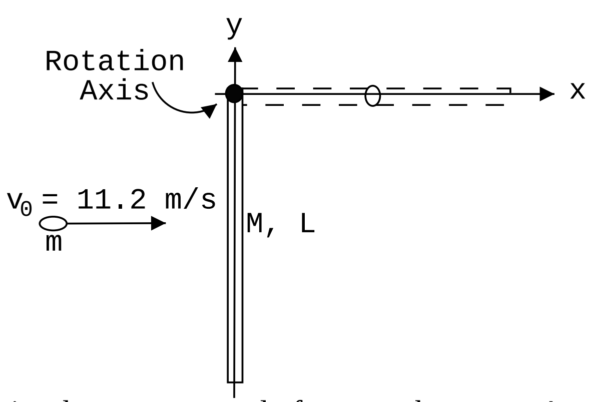
3. (e) If it's a frictionless bowling alley, the ball will always slide, since there is no torque on it to cause it to rotate.

4. (b) The original components of momentum are: $p_{x_i}: \quad mv\hat{i} + mv\cos\theta\hat{i}$; $p_{y_i}: \quad 0 + mv\sin\theta\hat{j}$. Since $p_i = p_f$, the components are equal afterwards, so (b) is correct.

5. (c) We can use conservation of energy to find this. $E_i = E_f$ yields: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. But, since the string does not slip, the tangential velocity of the wheel has to be equal to velocity of the falling block. This implies that $v = r\omega$ can be used to eliminate ω from our expression. Since $\omega = \frac{v}{R}$, the energy equation becomes $mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)\frac{v^2}{R^2}$ or $mgh = \frac{3}{4}mv^2$. From this, we find $|v| = \sqrt{\frac{4}{3}gh}$.

Problems

Problem II



A projectile of mass $m = 0.1$kg is fired horizontally with a velocity $v_0 = 11.2\frac{\mathrm{m}}{\mathrm{s}}$. at a rod of mass $M = 0.3$kg and an unknown length L. The projectile makes a perfectly inelastic collision with the rod, striking the center-of-mass of the rod.

Here is what we expected of you on the exam. A more correct solution is presented below.

a.) Use conservation of momentum to find the velocity (v_1) of recoil of the rod/projectile system, at the moment of impact.

$$mv_0 = (m+M)v_1 = (m+3m)v_1 = 4mv_1. \text{ So, } v_0 = 4v_1 \Rightarrow v_1 = \frac{1}{4}v_0 = \frac{11.2\frac{\mathrm{m}}{\mathrm{s}}}{4} = \boxed{2.8\frac{\mathrm{m}}{\mathrm{s}}}.$$

b.) Write down an expression for the total mechanical energy of the system at the moment of impact, in terms of m, g, L, and v_1. (Take $U(y = -\frac{1}{2}L) = 0$ where $y = 0$ is on the rotation axis).

$$E_{Mech,i} = K_i = \frac{1}{2}I\omega_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\frac{v_1^2}{\left(\frac{L}{2}\right)^2} = \frac{2}{3}(3m)v_1^2 + \frac{1}{2}mv_1^2 = \boxed{\frac{5}{2}mv_1^2}$$

c.) Similarly, write down the total mechanical energy when the rod/projectile system is instantaneously horizontal.

$$E_{Mech,f} = U_f = Mg\left(\tfrac{L}{2}\right) + mg\left(\tfrac{L}{2}\right) = (M+m)g\left(\tfrac{L}{2}\right) = 4mg\left(\tfrac{L}{2}\right) = \boxed{2mgL}$$

d.) Equate the expressions from parts (B) and (C) and solve for the length of the rod L.

$$E_{Mech,i} = E_{Mech,f} \Rightarrow \tfrac{5}{2}mv_1^2 = 2mgL \Rightarrow L = \tfrac{5v_1^2}{4g} = \tfrac{5\times\left(2.8\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{4\times\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2} = 1\mathrm{m}.$$

$\boxed{L = 1\mathrm{m}}$.

Alternate (and Correct!) Solution:

a.) Use conservation of (angular!) momentum to find the velocity (v_1) of recoil of the rod/projectile system, at the moment of impact.

To properly solve this part, use angular momentum (which had not yet been covered at test time). About the support point, $\vec{L}_i = \vec{L}_f$. Rotation is couter-clockwise about the axis through the support, so $L_i = \frac{L}{2}mv_0 = L_f$. In fact, despite what was asked in the problem, linear momentum can't be conserved because the pivot point of the rod exerts a horizontal force on the system to prevent the end of the rod from moving. This force creates no torque, however, since it's line of force passes through the pivot point.

$$\begin{aligned}
L_f &= I_{Total}\omega = (I_{rod} + I_m)\left(\frac{v_1}{\frac{L}{2}}\right) \\
&= \left[\frac{1}{3}ML^2 + m\left(\frac{L}{2}\right)^2\right]\left(\frac{v_1}{\frac{L}{2}}\right) = \left[\frac{1}{3}(3m)L^2 + m\left(\frac{L}{2}\right)^2\right]\left(\frac{v_1}{\frac{L}{2}}\right) \\
&= 2mL^2\frac{v_1}{L} + \frac{v_1}{2L}mL^2 = \frac{5}{2}mv_1L = L_i \\
\frac{1}{2}mv_0L &= \frac{5}{2}mv_1L \Rightarrow v_1 = \frac{1}{5}v_0 = \boxed{2.24\frac{\mathrm{m}}{\mathrm{s}}}
\end{aligned}$$

Compare this with the (improper) solution from linear momentum (as suggested in the problem) which results in $v_1 = \frac{1}{4}v_0 = 2.8\frac{\mathrm{m}}{\mathrm{s}}$.

b.) Write down an expression for the total mechanical energy of the system at the moment of impact, in terms of m, g, L, and v_1.

(Take $y = 0$ at the center of mass of the rod, then $U(y) = (m+M)gy = 0$ there).

$$E_{Mech,i} = K_i = \frac{1}{2}I\omega_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\frac{v_1^2}{\left(\frac{L}{2}\right)^2} + \frac{1}{2}mv_1^2 = \frac{2}{3}(3m)v_1^2 + \frac{1}{2}mv_1^2 = \boxed{\tfrac{5}{2}mv_1^2}.$$

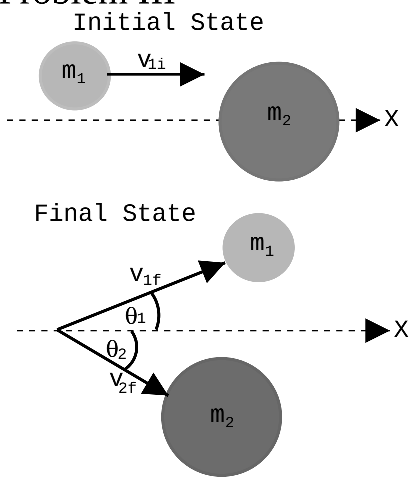
c.) Similarly, write down the total mechanical energy when the rod/projectile system is instantaneously horizontal.

$$E_{Mech,f} = U_f = Mg\left(\frac{L}{2}\right) + mg\left(\frac{L}{2}\right) = (M+m)g\left(\frac{L}{2}\right) = 4mg\left(\frac{L}{2}\right) = \boxed{2mgL}.$$

d.) Equate the expressions from parts (b) and (c) and solve for the length of the rod L.

$$E_{Mech,i} = E_{Mech,f} \Rightarrow \frac{5}{2}mv_1^2 = 2mgL \Rightarrow L = \frac{5v_1^2}{4g} = \frac{5\times\left(2.24\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{4\times\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2} = \boxed{0.64\mathrm{m}}.$$

Problem III



A collision occurs between two objects of unequal mass, $m_1 = 1\text{kg}$ and $m_2 = 4\text{kg}$. Mass m_2 is initially at rest, at the origin and m_1 is incident from the left with a velocity $\vec{v}_0 = 2\frac{\text{m}}{\text{s}}\hat{i}$. After the collision, mass m_1, has a velocity of magnitude $|\vec{v}_1| = 1\frac{\text{m}}{\text{s}}$ and emerges at an angle of $37°$.

a.) Find the initial momentum of the system.

$$\vec{p}_i = m_1 v_0 \hat{i} = (1\text{kg} \times 2\tfrac{\text{m}}{\text{s}})\hat{\text{i}} = \boxed{2\tfrac{\text{kg m}}{\text{s}}\hat{i}}.$$

b.) Write down the expression for the final momentum of the system.

$$\vec{p}_f = m_1 v_1 \cos(\theta_1)\hat{i} + m_1 v_1 \sin(\theta_1)\hat{j} + m_2 v_2 \cos(\theta_2)\hat{i} + m_2 v_2 \sin(\theta_2)\hat{j} = \boxed{(m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)\hat{i} + (m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2)\hat{j}}.$$

c.) By comparing (a) and (b), find the magnitude $|\vec{v}_2|$ and the angle θ_2 of object m_2 after the collision.

$2\frac{\text{kg m}}{\text{s}}\hat{i} = ((1)(1)(0.8 + (4)(v_2)\cos(\theta_2))\ \frac{\text{kg m}}{\text{s}}\hat{i}$
$0\frac{\text{kg m}}{\text{s}}\hat{j} = ((1)(1)(0.6) + 4v_2\sin(\theta_2))\ \frac{\text{kg m}}{\text{s}}\hat{j}$
$v_2\cos\theta_2 = 0.3$ and $v_2\sin\theta_2 = -0.15 \Rightarrow \tan\theta_2 = \frac{-0.15}{0.30} = -\frac{1}{2} \Rightarrow \theta_2 = -26.6°$.
Then $v_2 = \frac{0.3}{\cos(-26.6°)} = \frac{1}{3}\frac{\text{m}}{\text{s}}$.

$$\boxed{v_2 = \tfrac{1}{3}\tfrac{\text{m}}{\text{s}} \;\&\; \theta_2 = -26.6°}.$$

d.) Is the collision elastic? Justify your answer.

$K_i = \frac{1}{2}m_1 v_0^2 = \frac{1}{2}(1)(2)^2\text{J} = 2\text{J}$. $K_f = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2 = \frac{1}{2}(1)(1)^2 + \frac{1}{2}(4)\left(\frac{1}{3}\right)^2\text{J} = 0.72\text{J}$.
Therefore, since $K_i \neq K_f$ $\boxed{\text{the collision is } \textit{inelastic}}$.

Problem IV:

a.) Since they are interlocking, the linear accelerations at the point of contact must be the same, so $a_1 = a_2$. Since $a_i = r_i\alpha_i$, we know that $R\alpha = (R/2)\alpha_2$. So, $\boxed{\alpha_2 = 2\alpha}$

b.) Assume $\theta_0 \equiv 0$. Since the wheels start at rest, $\omega_0 = 0$ as well. So, for the larger wheel, we have $\boxed{\omega_1 = \alpha T}$. For the smaller wheel, we have $\boxed{\omega_2 = 2\alpha T}$.

c.) $W = \Delta KE = KE_f - KE_i = KE_f = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$. Substituting, we find

$$W = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha T)^2 + \frac{1}{2}\left(\frac{1}{2}M(R/2)^2\right)(2\alpha T)^2.$$

This simplifies to

$$W = \frac{1}{4}MR^2\alpha^2T^2 + \frac{1}{4}MR^2\alpha^2T^2 = \boxed{\tfrac{1}{2}MR^2\alpha^2T^2}$$

d.) To relate the torques on the two wheels, we note that the *forces* between the two wheels must be equal and opposite; call this force F. The torque on the smaller wheel is given by

$$\tau_2 = F(R/2) = I_2\alpha_2.$$

The torque on the larger wheel is given by

$$\tau_1 = \tau_{\text{cent}} - FR = I_1\alpha.$$

From the equation for the smaller wheel, we find

$$F(R/2) = \frac{1}{2}M(R/2)^2 \times 2\alpha = \frac{1}{4}MR^2\alpha; \;\; FR = \frac{MR^2}{2}\alpha.$$

Plugging this into the equation for the larger wheel, we find

$$\tau_{\text{cent}} - \frac{MR^2}{2}\alpha = \frac{1}{2}MR^2\alpha; \;\; \boxed{\tau_{\text{cent}} = mR^2\alpha}.$$

Problem V:

a.) Two quantities are conserved in this problem: the momentum in the horizontal plane and the total energy. Even though there is friction between the ramp and the rolling cylinder, the point of contact between the cylinder and the ramp is not sliding, so friction does no net work on the system ($F \cdot d = 0$ since $d = 0$). Momentum is conserved in the horizontal plane because there are no external horizontal forces on the system.

b.) Start with the energy equation first:

$$E_i = E_f \Rightarrow E_i = U_i = mgH; \quad E_f = KE_f = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I\omega_{\text{cm}}^2 + \frac{1}{2}MV^2$$

Here, H is measured from the height of the frictionless plane which is where $H = 0$. Note that we have to include the kinetic energy of the ramp! Since the cylinder rolls without slipping, $v_{\text{cm}} = R\omega_{\text{cm}}$, which we can substitute into the $\frac{1}{2}I\omega_{\text{cm}}^2$ term to remove ω from the equation. Also, for a cylinder, $I = \frac{1}{2}mR^2$.

c.) Using conservation of momentum,

$$p_{xi} = p_{xf} \;\; \Rightarrow 0 = MV + mv_{\text{cm}}, \;\; V = \frac{m}{M}v_{\text{cm}},$$

allowing us to relate V to $v_{\rm cm}$.

d.)

Collecting terms now, we can rewrite the energy equation in terms of only $v_{\rm cm}$.

$$\begin{aligned}
mgH &= \frac{1}{2}mv_{\rm cm}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\frac{v_{\rm cm}^2}{R^2} + \frac{1}{2}M\left(\frac{m^2}{M^2}v_{\rm cm}^2\right) \\
mgH &= \frac{1}{2}mv_{\rm cm}^2 + \frac{1}{4}mv_{\rm cm}^2 + \frac{1}{2}mv_{\rm cm}^2\left(\frac{m}{M}\right) \\
2gH &= v_{\rm cm}^2\left[\frac{3}{2} + \frac{m}{M}\right] \\
v_{\rm cm} &= \sqrt{\frac{2gH}{\frac{3}{2}+\frac{m}{M}}}.
\end{aligned}$$

Going back to the momentum equation, we can then find V:

$$V = \frac{m}{M}v_{\rm cm};\ \ V = \frac{m}{M}\sqrt{\frac{2gH}{\frac{3}{2}+\frac{m}{M}}}$$