I. Multiple Choice Questions

1.) $\boxed{\text{c)}}$ Regardless of the magnitude of the gravitational acceleration or whether or not the trajectory leads to maximum range, if there is no acceleration in the horizontal direction, the horizontal velocity is constant.

2.) $\boxed{\text{d)}}$ If the translational and rotational kinetic energies are equal, $E_i = E_f$ implies that

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mv^2 \ \Rightarrow \ h = v^2/g.$$

3.) $\boxed{\text{a)}}$ $\omega_0 = \sqrt{\frac{k}{m}}$. If $m \to 4m$, $\omega = \sqrt{\frac{k}{4m}} = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}w_0$.

4.) $\boxed{\text{c)}}$ The period has units of seconds, so we need a combination of quantities that combine to give us this. Of the physical parameters in this problem, I, M, D, and g, only g has seconds as part of its dimensions ($g = m/s^2$). So, $\sqrt{D/g}$ would have dimensions of seconds, but this is the expression for a simple pendulum, and our object is more complicated. Note that I/M has units of m^2, since $I \propto MR^2$. So, $I/(MgD)$ has units of s^2 since we have two units of meters each in numerator and denominator. The quantity $\sqrt{I/(MgD)}$ is the only choice, then.

5.) $\boxed{\text{c)}}$ Gravity exerts an external torque on the ball, causing its angular momentum to precess (the angular momentum has a vertical component and a horizontal one with rotates around the center). Also, the force of the string pulling on the ball does not act through the center of the pole, but off-axis, so there is a net tangential force on the ball from the string, too. Both of these forces contribute to the non-conservation of angular momentum.

6.) $\boxed{\text{b)}}$ For closed orbits, the total energy must be negative if we define the gravitation potential to be zero at $r = \infty$.

7.) $\boxed{\text{b)}}$ This is a Newton's law problem:

$$\begin{aligned}\sum F_x &= -T_B \cos 53^\circ + T_C \cos 37^\circ = 0 \\ \sum F_y &= T_B \sin 53^\circ + T_C \sin 37^\circ - 10 \text{ N} = 0\end{aligned}$$

Using the top equation, we find that $T_B = T_C \cos 37^\circ / \cos 53^\circ = 1.32 T_C$. Substituting this into the second equation, we can solve for T_C. $1.66 T_C = 10$ N, $T_C = 10$ N.

8.) $\boxed{\text{b)}}$ At the equlibrium point, the force of the spring just balances that of gravity, so $\sum F_y = 0 = -kY_f - mg \Rightarrow Y_f = -mg/k$.

Problem II (16 points)

a) The moment of inertia of each gondola is given by MR^2, so the total moment of inertia is $4MR^2 = 4 \times 500 \times 25 = 50,000\ \mathrm{kg\,m^2}$.
b) $\tau = I\alpha = 50,000\ \mathrm{kg\,m^2} \times 0.05\mathrm{rad/s} = 2500\ \mathrm{N\,m}$.
c) Stress is put on the cables because they have to provide the centripetal force necessary to keep the gondolas moving in a circle. The maximum tangential velocity each of the cables can support can be found by equating the centripetal forces:

$$mv_{max}^2/R = F_{max} = 200,000\ N \Rightarrow v_{max} = 44.7\mathrm{m/s}.$$

Since we know the angular acceleration and the relation $v = r\omega$ between tangential and rotational velocities, we can use the rotational kinematic equations with these inputs: $\omega_i = 0$, $\omega_f = v_{max}/R = 8.94\ \mathrm{rad/s}$, and $\alpha = 0.05\ \mathrm{rad/s^2}$. Using

$$\omega_f = \omega_i + \alpha t \Rightarrow t = 178.9\mathrm{s}.$$

d) In terms of V, we can write down a relationship between the energy lost to friction and the distance travelled (D):

$$\frac{1}{2}mV^2 = \mathcal{F}_k \times D,\ \ D = \frac{1}{2}mV^2/\mathcal{F}_k,\ \ \mathcal{F}_k = \mu\mathcal{N} = \mu mg\ \ \Rightarrow D = \frac{1}{2}V^2/\mu g$$

Plugging in numbers ($V = v_{max}$), $D = 145.6$ m.

Problem III: (15 points)

a) The velocity of m_1 can be found using conservation of energy. If we set $h = 0$ at the floor, then

$$E_i = E_f\ \Rightarrow\ m_1gh = m_2gh + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2.$$

Both of the masses have the same velocity because they are connected by the string. We want to find v:

$$gh(m_1 - m_2) = \frac{1}{2}v^2(m_1 + m_2);\ \ v^2 = \frac{2gh(m_1 - m_2)}{m_1 + m_2};\ \ v = \sqrt{\frac{2gh(m_1 - m_2)}{m_1 + m_2}}.$$

b) When mass 1 hits the ground, mass 2 will continue to rise, since it is travelling upwards with the velocity v we found in (a). Probably the easiest way to find out how high it goes is to find the height above h that it travels in this second part of the motion, then add this to h to get the total height. We can find the extra height (call it d) by conservation of energy, too:

$$E_i = E_f\ \Rightarrow\ \frac{1}{2}m_2v^2 = m_2gd.$$

Here, we have set the potential to be zero at $h = h$. Solving for d, we find:

$$\frac{1}{2}\frac{2gh(m_1 - m_2)}{m_1 + m_2} = gd,\ \ d = \frac{h(m_1 - m_2)}{m_1 + m_2}.$$

This gives us the total height H reached by mass m_2 as

$$H = h + d = h + \frac{h(m_1 - m_2)}{m_1 + m_2} = h\left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right)$$

c) Mass m_1 goes from an initial velocity of $v_0 = 0$ to a final velocity of $v_f = v$ over the distance h, so we can use standard kinematic equations to find its acceleration:

$$v_f^2 = v_0^2 + 2ah \ \Rightarrow\ v_f^2 = 2ah \ \Rightarrow\ a = \frac{v_f^2}{2h} = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right),$$

which is what we would expect from an Atwood's machine.
d) The acceleration would be *smaller*. Some force is required to spin the wheel, so that the net force on m_2 is decreased, or, correspondingly, the "effective mass" of the system is increased, leading to a smaller acceleration. OR: Some of the gravitational potential energy of mass m_1 now must appear as rotational energy of the wheel, so the final velocity of m_1 will be smaller, which implies that the acceleration is smaller.

Problem IV: (15 points)

a) The angular frequency is given by $\omega = \sqrt{\frac{k}{M}}$, and the period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{k}}$.
b) If we take $x = -A\cos(\omega t)$ as the solution for the position, we can see that at $t = 0$,
$x = -A\cos(\omega \times 0) = -A$, so it gives the correct initial position.
c) The velocity is given by $v = \frac{dx}{dt} = \omega A \sin(\omega t)$, which, at $t = 0$, is zero, also satisfying the initial conditions.
d) The acceleration is given by $a = \frac{dv}{dt} = \omega^2 A\cos(\omega t)$, which, at $t = 0$, is $a = \omega^2 A$. Using $\omega = \sqrt{\frac{k}{M}}$, we can rewrite this as $a = \frac{k}{M}A$. Since we know that the force at $t = 0$ is given by $F = -kx = -k(-A) = Ma$, we can see that the initial acceleration has to be $a = \frac{k}{M}A$, so this checks out as well.
e) Travelling from the maximum amplitude to the equilibrium position takes a quarter-period, so $t = T/4 = \frac{\pi}{2}\sqrt{\frac{M}{k}}$.

Problem V: (15 points)

a) Take the more massive star to be the origin of our coordinate system. The position of the center of mass is then found from:

$$X_{\text{c.m.}} = \frac{(3M)\times(0) + (M)\times(D)}{3M + M} = \frac{1}{4}D$$

b) The solution to part (a) implies that the radius R_1 of the orbit of the lighter star is $\frac{3}{4}D$, so we can use this to equate the centripetal force to the gravitational force in order to solve for the star's velocity: (Note that the distance between the stars is D)

$$\sum F_r = \frac{MV^2}{R_1} = \frac{GM(3M)}{D^2};\ \ V^2 = \frac{3GM}{D^2}\frac{3}{4}D = \frac{9GM}{4D} \ \Rightarrow\ V = \frac{3}{2}\sqrt{\frac{GM}{D}}.$$

c) The period of the motion is just given by

$$T = \frac{2\pi R_1}{V} = \frac{2\pi\frac{3}{4}D}{\frac{3}{2}\sqrt{\frac{GM}{D}}} = \frac{\pi D^{3/2}}{\sqrt{GM}}.$$

Problem VI: (15 points)

a.) What is the initial momentum, $\vec{p}$, of the system?

$p_x = 0.02\text{kg} \times 1.6\ \frac{\text{m}}{\text{s}} \times \cos(45°) = 0.023\ \frac{\text{kg m}}{\text{s}}$ $\quad p_y = -0.02\text{kg} \times 1.6\ \frac{\text{m}}{\text{s}} \times \sin(45°) = -0.023\ \frac{\text{kg m}}{\text{s}}$

$\boxed{\vec{p} = 0.023\ \frac{\text{kg m}}{\text{s}}(\hat{x} - \hat{y})}$.

b.) What is the final momentum, $\vec{p}\,'$, of the system?

$p'_{A,x} = (0.02\text{kg})(1\ \frac{\text{m}}{\text{s}})\cos(20°) = 0.019\ \frac{\text{kg m}}{\text{s}}$. $\quad p'_{A,y} = (0.02\text{kg})(1\ \frac{\text{m}}{\text{s}})\sin(20°) = 0.0068\ \frac{\text{kg m}}{\text{s}}$.

$$\boxed{\vec{p}\,' = (0.019\ \frac{\text{kg m}}{\text{s}} + 0.04\ \text{kg} \times v_{B,x})\hat{x} + (0.0068\ \frac{\text{kg m}}{\text{s}} + 0.04\ \text{kg} \times v_{B,y})\hat{y}}$$

or, in terms of θ and v_B:

$$\boxed{\vec{p}\,' = (0.019\ \frac{\text{kg m}}{\text{s}} + 0.04\ \text{kg} \times v_B \sin\theta)\hat{x} + (0.0068\ \frac{\text{kg m}}{\text{s}} - 0.04\ \text{kg} \times v_B \cos\theta)\hat{y}}$$

c.) What is the speed and direction of puck B after the collision?

$p_x = p_x^{'} \Rightarrow 0.023\ \frac{\text{kg m}}{\text{s}} = 0.019\ \frac{\text{kg m}}{\text{s}} + 0.04\text{kg} \times v_{B,x} \Rightarrow v_{B,x} = \frac{0.023\ \frac{\text{kg m}}{\text{s}} - 0.019\ \frac{\text{kg m}}{\text{s}}}{0.04\ \text{kg}} = 0.1\ \frac{\text{m}}{\text{s}}$.

$p_y = p_y^{'} \Rightarrow -0.023\ \frac{\text{kg m}}{\text{s}} = 0.068\ \frac{\text{kg m}}{\text{s}} + 0.04\ \text{kg} \times v_{B,y} \Rightarrow v_{B,y} = \frac{-0.023\ \frac{\text{kg m}}{\text{s}} - 0.0068\ \frac{\text{kg m}}{\text{s}}}{0.04\ \text{kg}} = -0.75\ \frac{\text{m}}{\text{s}}$.

speed of puck B = $|v_B| = \sqrt{v_{B,x}^2 + v_{B,y}^2} = \boxed{0.76\ \frac{\text{m}}{\text{s}}}$.

As shown on the diagram, $\tan\theta = \frac{|v_{B,x}|}{|v_{B,y}|}$, $\theta = \tan^{-1}\left(\frac{0.1\ \frac{\text{m}}{\text{s}}}{0.75\ \frac{\text{m}}{\text{s}}}\right) = \boxed{7.6°}$.