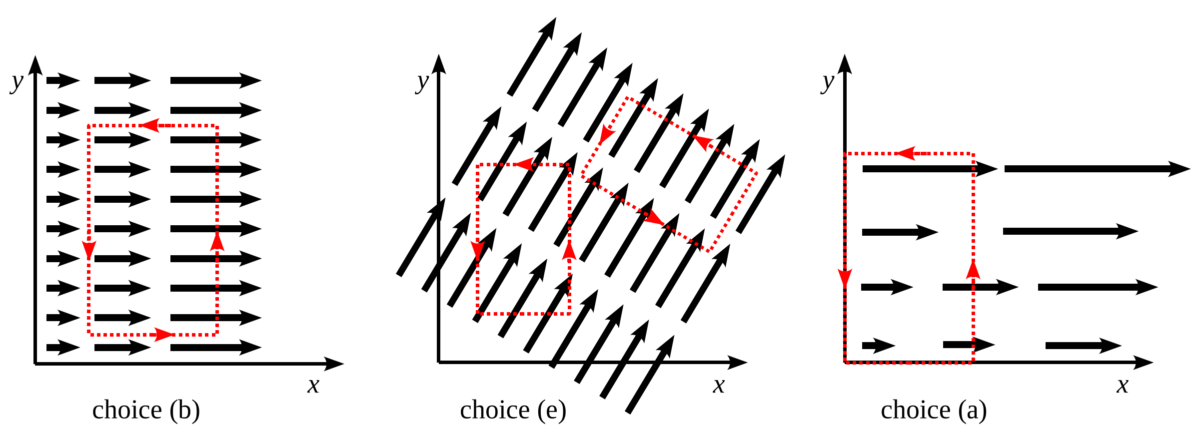
Physics 10411 Fall 2008 Final Exam Solutions

I. Multiple Choice

1. (b) One way to do this is with Conservation of Energy. The work done by the force is equal the change in kinetic energy: $0 - \frac{1}{2}mv_0^2 = -F \cdot d$. Solving for m, we find $m = \frac{2FD}{v_0^2}$. Plugging in numbers, $m = 3.0$ kg.

2. (e) There are a couple of steps to this one. First, we need to relate the velocity at the bottom of the swing to the height: $U_i = KE_f \Rightarrow mgh = \frac{1}{2}mv^2 \Rightarrow v^2 = 2gh$. Then, we have an *inelastic collision*. Since it is inelastic, energy is *not* conserved. Conserving momentum, $mv = 3mV \Rightarrow V = \frac{1}{3}v$. Then, the combined mass swings up again solely under the influence of gravity, so we conserve energy again: $\frac{1}{2}(3m)V^2 = 3mgH \Rightarrow H = V^2/2g$. We know from the first parts that $V = \frac{1}{3}v$ and what v is in terms of h, so we just have to plug in: $H = (\frac{1}{3}v)^2/2g = \frac{1}{9}h$.

3. (a) The hint basically tells you what to do. If you draw closed rectangles or parallelograms in the $x - y$ plane, the work done by the force $\vec{\mathbf{F}}$ should be zero if you go all the way around. Any of the forces that are either constant, or only depend on the distance from one of the coordinate axes are conservative, since you can draw a box like the one shown in the figures for choices (b) or (e), below. Here, the length of the arrows at a given point indicate the magnitude of the force at that point in the plane. You can see that the work done moving perpendicular to the force is zero, and since the value of the force is the same at any point on the other two sides, you get equal and opposite work when traversing the two sides in opposite directions. This is also true for the rectangle in choice (e) that isn't perpendicular to the forces: you still get equal and opposite work for the opposite sides of the rectangle. For choice (a), though, you have the situation shown on the far right. The force is zero on the x and y axes, so three sides of the box give zero work. The top side, however, is non-zero, so this is not a conservative force.



choice (b) choice (e) choice (a)

4. (b) Projectile motion. We just need to relate the range to the initial velocity of the projectile. To do this, it's easiest to find the time of flight, then compute the range. We have the usual relationship

$$v_f = v_i + at \Rightarrow -v_{0y} = v_{0y} - gt \quad \text{where we have defined up to be } + .$$

This allows us to find the time when the projectile has returned to the ground, since its y velocity will be equal and opposite to its original y velocity at this point. Solving, we find $t = 2v_{0y}/g$, which, when using $\sin(45°) = \frac{\sqrt{2}}{2}$, gives $t = \sqrt{2}v_0/g$. Next, we want to use this time to calculate the range: $x = v_{0x}t = v_0 \cos(45°)t = v_0^2/g$. So, we just have to put in numbers, converting 60 km to 60,000 m, and solving for v_0. This gives $v_0 = 767\text{m/s}$.

5. (d) Assume the mass of the disk is M. The mass density per area σ is given by $\sigma = \frac{M}{\pi R^2}$. Using this, we can figure out how much mass is removed when we slice out the hole: $m = M\frac{\pi(R/2)^2}{\pi R^2} = \frac{1}{4}M$. So, the total mass of the remainder is $\frac{3}{4}M$. We can use this now to write an equation for the position of the center of mass, defining our origin at the bottom of the original disk:

$$y_{CM} = \frac{MR - m(R/2)}{\frac{3}{4}M}.$$

We've included a minus sign for the missing mass since it's not there anymore. Solving, we find $\frac{7}{8}R \times \frac{4}{3} = \frac{7}{6}R$, so the center of mass moves upward by $\frac{1}{6}R$.

Problems

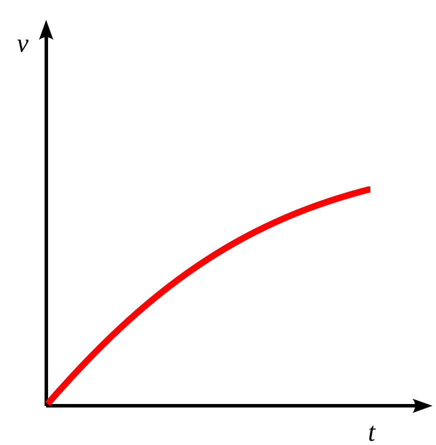
II. (a) We can use the equation

$$P = Fv = mav = m\frac{dv}{dt}v \quad \Rightarrow \quad \frac{P}{m}dt = v\,dv. \quad \text{Integrating,}$$

$$\frac{P}{m}\int_0^t dt = \int_0^{v_f} v\,dv \quad \Rightarrow \quad \frac{P}{m}t = \frac{1}{2}v_f^2.$$

We can go one step further and write $v = \sqrt{\frac{2P}{m}t}$. Note that acceleration is not constant, so none of the standard kinematic equations work properly. You could, however, have just noted that the work done was $\frac{1}{2}mv^2$ during the time interval, so $P = \frac{1}{2}mv^2/t$. This gives the same answer.

(b) A sketch of v vs. t would look something like this:



What does it tell us about the acceleration? Clearly, it's not constant, and it decreases with time. You can explain this in a number of different ways. Since $P = dW/dt$, then power tells us about the rate of change of the kinetic energy of the dragster. This implies, as our equation says, that a graph of v^2 vs time would be a straight line, since the rate of increase of kinetic energy is constant. Our graph of v vs. t rolls over because it takes a smaller increment in v to keep v^2 linear in time when v is larger.

(c) We can use our result from (a) to solve for P:

$$P = \frac{\frac{1}{2}mv_f^2}{t} = 2.76 \text{ MW!}$$

Lots of power! Real measured outputs for dragsters are in the 4.5-6 MW range, so we're within a factor of two. Not bad, really. Clearly, in real life a lot of power is spent overcoming drag and other friction sources, so our estimate that it all goes into kinetic energy is clearly optimistic.

(d) BONUS: Since we have an expression for $v(t)$, we can easily do one more integral to relate this to $x(t)$:

$$v = \frac{dx}{dt} = \sqrt{\frac{2P}{m}} t^{\frac{1}{2}} \quad \Rightarrow \quad dx = \sqrt{\frac{2P}{m}} t^{\frac{1}{2}} dt \quad \Rightarrow \quad \int_0^{x_f} dx = \sqrt{\frac{2P}{m}} \int_0^t t^{\frac{1}{2}} dt.$$

Solving this yields

$$x_f = \sqrt{\frac{2P}{m}} \frac{2}{3} t^{\frac{3}{2}}. \quad \text{Squaring and solving for } P : P = \frac{9}{8} \frac{mx^2}{t^3}.$$

Plugging in some numbers here gives $P = 2.1$ MW. Still a lot of power! How do we reconcile the differences in these calculations? Clearly, the real dragster has gears, so there is not constant torque applied to the engine. We're looking at average quantities with a strange assumption that the overall power is constant. So it's not surprising that by looking at velocity with time and distance with time that we get slightly different answers.

III. (a) Since the string is purely a radial force, there is no torque in this problem, so angular momentum is conserved. $L_i = \vec{r} \times \vec{p} = Rmv$ since $\vec{r} \perp \vec{v}$. $L_f = rmv_f$, so we can set $L_i = L_f$ to find a relationship between the new and old speeds:

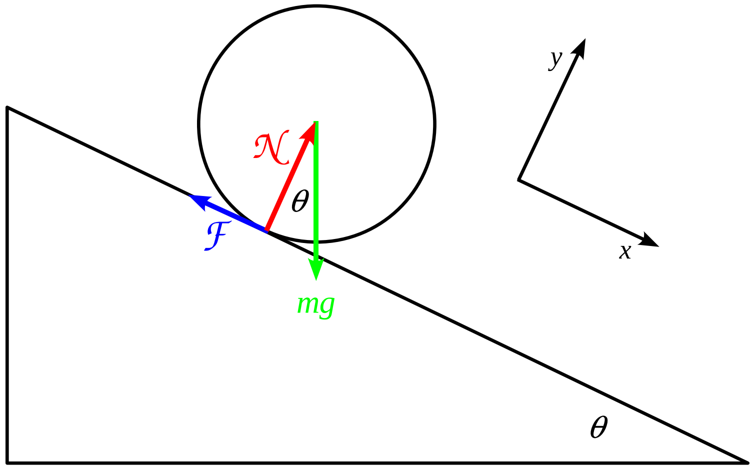
$$Rmv = rmv_f \quad \Rightarrow \quad v_f = \frac{R}{r} v.$$

(b) Since this is a frictionless horizontal table, the only force in the radial direction is the tension in the string:

$$\sum F_r = \frac{mv_f^2}{r} = T = m\left(\frac{R}{r}v\right)^2 / r = \frac{mv^2R^2}{r^3}.$$

(c) $W = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv^2 = \frac{1}{2}mv^2\frac{R^2}{r^2} - \frac{1}{2}mv^2 = \frac{1}{2}mv^2\left(\frac{R^2}{r^2} - 1\right).$

IV. (a) The free body diagram should have the forces $\mathcal{F}_s$, $\mathcal{N}$, and the weight of the sphere, mg, as shown on the figure. The coordinate axes we will use in the solution are also indicated.



(b) Newton's equations for translation and rotation should look like

$$\begin{aligned}\sum F_x &= ma_x = mg\sin\theta - \mathcal{F}_s \\ \sum F_x &= 0 = \mathcal{N} - mg\cos\theta \\ \sum \tau_{CM} &= I_{CM}\alpha_{CM} = \mathcal{F}_s R\end{aligned}$$

(c) To determine the acceleration of the center of mass, we need to relate the linear and angular accelerations for rolling without slipping, $a_x = R\alpha$. We can then use this to solve for the force of friction in the torque equation in terms of the linear acceleration:

$$\mathcal{F}_s R = I_{CM}\frac{a_x}{R} \quad \Rightarrow \quad \mathcal{F}_s = I_{CM}\frac{a_x}{R^2}.$$

[Note that $\mathcal{F}_s \neq \mu_s\mathcal{N}$! We don't know what the coefficient of friction is, and the answer shouldn't depend on it. You have to use the torque equation to eliminate friction from the problem.] We can then substitute the above result into the x force equation:

$$ma_x = mg\sin\theta - I_{CM}\frac{a_x}{R^2} \quad \Rightarrow \quad a_x\left(m + \frac{I_{CM}}{R^2}\right) = mg\sin\theta$$

Solving for a_x and using the fact that $I_{CM} = \frac{2}{3}mR^2$ for a thin spherical shell, we find:

$$a_x = \frac{mg\sin\theta}{m + \frac{I_{CM}}{R^2}} = \frac{mg\sin\theta}{m + \frac{2}{3}m} = \frac{3}{5}g\sin\theta.$$

In principle, one could do this problem with conservation of energy by figuring out what the velocity of the rolling shell is at the bottom given some starting height. You could then figure out what the linear accleration is given the distance it has traveled. This should give the same answer.

V. (a) This is a Conservation of Energy problem. We can find the compression of the spring by relating the initial and final energies: $E_i = E_f$. At the beginning of the problem, the block is at rest, and at the point of maximum compression, the block is also at rest, so we merely have to keep track of potential energy. We will choose the zero of gravitational potential to be the point where the block hits the spring. The block starts a height $d\sin\theta$ above this point, and ends up a height $x \sin\theta$ below this point, where x is the compression of the spring, and $d = 4$ meters. Our energy equation is then

$$E_i = E_f \quad \Rightarrow \quad U_i = U_f \quad \Rightarrow \quad mgd\sin\theta = -mgx\sin\theta + \frac{1}{2}kx^2.$$

This is a quadratic equation in x, the only unknown in the problem. Putting in numbers and solving for x yields $x = 0.99$ meters.

(b) If friction is added, we have the additional complication that friction takes work out of the system, so our energy equation really reads $\Delta E_{mech} = W_{\mathcal{F}}$. We can keep the spring and gravity as "internal", in which case we have a similar situation to part (a): $\Delta E_{mech} = U_f - U_i = -mgx\sin\theta + \frac{1}{2}kx^2 - mgd\sin\theta$. This must be equal to the work done by friction: $W_{\mathcal{F}} = -\mathcal{F}_k(d+x)$. We can also include some energy lost to heat. So, finally, we have

$$-mgx\sin\theta + \frac{1}{2}kx^2 - mgd\sin\theta = -\mathcal{F}_k(d+x) + \Delta E_{\text{int}}$$

Re-arranging, it's clearer to see where the energy goes if we write

$$-mgx\sin\theta + \frac{1}{2}kx^2 = mgd\sin\theta - \mathcal{F}_k(d+x) + \Delta E_{\text{int}}.$$

The right hand side has the initial energy of the system plus that dissipated by friction, leaving the energy content on the left. To solve, one needs the magnitude of the friction force, which comes from $\sum F_y = 0 \Rightarrow \mathcal{N} = mg\cos\theta$.

VI. (a) Conservation of momentum: $mv = MV + \frac{1}{2}mv \Rightarrow V = \frac{m}{M}\frac{v}{2}$. Note: there is no guarantee that energy is conserved. Momentum *has to be* conserved in a collision in which there are only internal forces between the objects. If you calculate how much energy is in the problem after the bullet passes through using the result here, you'll find that some energy is actually lost as the bullet passes.

(b) For our warrior to just make it all of the way around the circle, it must be the case that the string just stays taught at the top. This implies that the tension is instantaneously zero at the very top, leaving the weight to provide the only centripetal force:

$$\sum F_r = \frac{Mv_T^2}{\ell} = T + Mg = Mg \quad \Rightarrow \quad \frac{1}{2}Mv_T^2 = \frac{Mg\ell}{2}.$$

where v_T is the velocity of the warrior at the top. Now, we can use conservation of energy to relate the speed at the top to the speed immediately after the collision. Taking the zero of the gravitational potential energy to be at the bottom of the swing, we can write

$$KE_i = KE_f + U_f \quad \Rightarrow \quad \frac{1}{2}MV^2 = \frac{1}{2}Mv_T^2 + Mg(2\ell) = \frac{1}{2}Mg\ell + 2Mg\ell = \frac{5}{2}Mg\ell$$

So, we only have to substitute in our original relationship between v and V and we're done:

$$V^2 = \left(\frac{m}{M}\frac{v}{2}\right)^2 = 5g\ell \quad \Rightarrow \quad v = \frac{M}{m}\sqrt{20g\ell}.$$

(c) If we replace the string with a rod, then angular momentum about the pivot would be conserved instead of linear momentum. Conservation of energy equations would have to worry about the change in height of the center of mass of the rod as well as the warrior. The velocity at the top would go to zero in order for a complete circle to barely be possible.