"Moderated Mediation Analysis Using Bayesian Methods"

by Lijuan Wang and Kristopher J. Preacher (2014)

Appendix A

Testing whether a mediation effect is moderated using the delta method

This appendix contains hypotheses and derivations of point estimation ($f(\hat{\boldsymbol{\theta}})$) and standard error ($s.e.(f(\hat{\boldsymbol{\theta}})$) for testing whether a mediation effect is moderated for the 6 scenarios presented in Table 1. The first-order and second-order variance estimates of the point estimate based on the delta method can be derived by $\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}$ and $\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D} + \frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}$ (see Preacher et al., 2007), where $\boldsymbol{D} = \partial_{\boldsymbol{\theta}}(f(\hat{\boldsymbol{\theta}}))$ and $\boldsymbol{H} = \partial^2_{\boldsymbol{\theta}}(f(\hat{\boldsymbol{\theta}}))$. Therefore, the first-order test statistic is $Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}}}$ and the second-order test statistic is $Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}}}$.

Scenario 0: no moderation

The null hypothesis is $H_0 : ab = 0$ with a point estimate of $f(\hat{\theta}) = \hat{a}\hat{b}$. The first-order and second-order variance estimates of the point estimate are $\hat{b}^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}}$ and $\hat{b}^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}} + s^2_{\hat{a}} s^2_{\hat{b}}$. Therefore, the first-order test statistic is $Z = \frac{\hat{a}\hat{b}}{\sqrt{\hat{b}^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}}}}$ and the second-order test statistic is $Z = \frac{\hat{a}\hat{b}}{\sqrt{\hat{b}^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}} + s^2_{\hat{a}} s^2_{\hat{b}}}}$.

Scenario 1: b is moderated by the input variable X

The null hypothesis is $H_0 : ab_2(x_1 - x_2) = 0$ with a point estimate of $f(\hat{\theta}) = \hat{a}\hat{b}_2(x_1 - x_2)$. The first-order and second-order variance estimates of the point estimate are $(\hat{b}_2^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}_2})(x_1 - x_2)^2$ and $(\hat{b}_2^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}_2} + s^2_{\hat{a}} s^2_{\hat{b}_2})(x_1 - x_2)^2$. Therefore, the first-order test statistic is $Z = \frac{\hat{a}\hat{b}_2}{\sqrt{\hat{b}_2^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}_2}}}$ and the second-order test statistic is $Z = \frac{\hat{a}\hat{b}_2}{\sqrt{\hat{b}_2^2 s^2_{\hat{a}} + \hat{a}^2 s^2_{\hat{b}_2} + s^2_{\hat{a}} s^2_{\hat{b}_2}}}$.

Scenario 2: a is moderated by a moderator variable Z but b is not moderated by Z

The null hypothesis is $H_0 : a_2 b(z_1 - z_2) = 0$ where z_1 and $z_2$are two values of the moderator variable Z. The point estimate is $\hat{a}_2\hat{b}(z_1 - z_2)$ and the first-order and second-order variance estimates of the point estimate are $(\hat{a}_2^2 s^2_{\hat{b}} + \hat{b}^2 s^2_{\hat{a}_2})(z_1 - z_2)^2$ and

$(\hat{a}_2^2 s_{\hat{b}}^2 + \hat{b}^2 s_{\hat{a}_2}^2 + s_{\hat{a}_2}^2 s_{\hat{b}}^2)(z_1 - z_2)^2$. Therefore, the first-order test statistic is $Z = \frac{\hat{a}_2\hat{b}}{\sqrt{\hat{a}_2^2 s_{\hat{b}}^2 + \hat{b}^2 s_{\hat{a}_2}^2}}$ and the second-order test statistic is $Z = \frac{\hat{a}_2\hat{b}}{\sqrt{\hat{a}_2^2 s_{\hat{b}}^2 + \hat{b}^2 s_{\hat{a}_2}^2 + s_{\hat{a}_2}^2 s_{\hat{b}}^2}}$.

Scenario 3: b is moderated by Z but a is not moderated by Z

The null hypothesis is $H_0 : ab_2(z_1 - z_2) = 0$ where z_1 and z_2 are two values of the moderator variable Z. The point estimate is $\hat{a}\hat{b}_2(z_1 - z_2)$ and the first-order and second-order variance estimates of the point estimate are $(\hat{a}^2 s_{\hat{b}_2}^2 + \hat{b}_2^2 s_{\hat{a}}^2)(z_1 - z_2)^2$ and $(\hat{a}^2 s_{\hat{b}_2}^2 + \hat{b}_2^2 s_{\hat{a}}^2 + s_{\hat{a}}^2 s_{\hat{b}_2}^2)(z_1 - z_2)^2$. Therefore, the first-order test statistic is $Z = \frac{\hat{a}\hat{b}_2}{\sqrt{\hat{a}^2 s_{\hat{b}_2}^2 + \hat{b}_2^2 s_{\hat{a}}^2}}$ and the second-order test statistic is $Z = \frac{\hat{a}\hat{b}_2}{\sqrt{\hat{a}^2 s_{\hat{b}_2}^2 + \hat{b}_2^2 s_{\hat{a}}^2 + s_{\hat{a}}^2 s_{\hat{b}_2}^2}}$.

Scenario 4: a is moderated by Z_1 and b is moderated by Z_2

The null hypothesis is $H_0 : a_1 b_2(z_{21} - z_{22}) + a_2 b_1(z_{11} - z_{12}) + a_2 b_2(z_{11}z_{21} - z_{12}z_{22}) = 0$ where z_{11} and z_{12} are two values of the moderator variable Z_1 and z_{21} and z_{22} are two corresponding values of the moderator variable Z_2. Here, we have

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \, \hat{a}_2 \, \hat{b}_1 \, \hat{b}_2]$$

$$f(\hat{\boldsymbol{\theta}}) = \hat{a}_1\hat{b}_2(z_{21} - z_{22}) + \hat{a}_2\hat{b}_1(z_{11} - z_{12}) + \hat{a}_2\hat{b}_2(z_{11}z_{21} - z_{12}z_{22})$$

$$\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & s_{\hat{a}_1\hat{a}_2} & 0 & 0 \\ s_{\hat{a}_1\hat{a}_2} & s_{\hat{a}_2}^2 & 0 & 0 \\ 0 & 0 & s_{\hat{b}_1}^2 & s_{\hat{b}_1\hat{b}_2} \\ 0 & 0 & s_{\hat{b}_1\hat{b}_2} & s_{\hat{b}_2}^2 \end{bmatrix}$$

and thus

$$\begin{aligned}\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D} = {} & s_{\hat{b}_1}^2 \hat{a}_2^2 (z_{11} - z_{12})^2 + s_{\hat{a}_1}^2 \hat{b}_2^2 (z_{21} - z_{22})^2 + 2(\hat{a}_1\hat{a}_2 s_{\hat{b}_1\hat{b}_2} + \hat{b}_1\hat{b}_2 s_{\hat{a}_1\hat{a}_2})(z_{11} - z_{12})(z_{21} - z_{22}) \\ & + 2\hat{a}_2^2 s_{\hat{b}_1\hat{b}_2}(z_{11} - z_{12})(z_{11}z_{21} - z_{12}z_{22}) + 2\hat{b}_2^2 s_{\hat{a}_1\hat{a}_2}(z_{21} - z_{22})(z_{11}z_{21} - z_{12}z_{22}) \\ & + s_{\hat{b}_2}^2(\hat{a}_1(z_{21} - z_{22}) + \hat{a}_2(z_{11}z_{21} - z_{12}z_{22}))^2 + s_{\hat{a}_2}^2(\hat{b}_1(z_{11} - z_{12}) + \hat{b}_2(z_{11}z_{21} - z_{12}z_{22}))^2\end{aligned}$$

$$\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\} = s^2_{\hat{a}_1}s^2_{\hat{b}_2}(z_{21}-z_{22})^2 + s^2_{\hat{a}_2}s^2_{\hat{b}_1}(z_{11}-z_{12})^2 + s^2_{\hat{a}_2}s^2_{\hat{b}_2}(z_{11}z_{21}-z_{12}z_{22})^2$$
$$+ 2s_{\hat{a}_1\hat{a}_2}s^2_{\hat{b}_2}(z_{11}z_{21}-z_{12}z_{22})(z_{21}-z_{22}) + 2s_{\hat{b}_1\hat{b}_2}s^2_{\hat{a}_2}(z_{11}z_{21}-z_{12}z_{22})(z_{11}-z_{12})$$
$$+ 2s_{\hat{a}_1\hat{a}_2}s_{\hat{b}_1\hat{b}_2}(z_{11}-z_{12})(z_{21}-z_{22})$$

Therefore, the first-order test statistic is $Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}}}$ and the second-order test statistic is $Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}}}$ accordingly.

Scenario 5: a and b are both moderated by Z

The null hypothesis is $H_0 : (a_1b_2 + a_2b_1)(z_1 - z_2) + a_2b_2(z_1^2 - z_2^2) = 0$ where z_1 and z_2 are two values of the moderator variable Z. For this scenario, we have

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1\,\hat{a}_2\,\hat{b}_1\,\hat{b}_2]$$

$$f(\hat{\boldsymbol{\theta}}) = (\hat{a}_1\hat{b}_2 + \hat{a}_2\hat{b}_1)(z_1 - z_2) + \hat{a}_2\hat{b}_2(z_1^2 - z_2^2)$$
$$\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s^2_{\hat{a}_1} & s_{\hat{a}_1\hat{a}_2} & 0 & 0 \\ s_{\hat{a}_1\hat{a}_2} & s^2_{\hat{a}_2} & 0 & 0 \\ 0 & 0 & s^2_{\hat{b}_1} & s_{\hat{b}_1\hat{b}_2} \\ 0 & 0 & s_{\hat{b}_1\hat{b}_2} & s^2_{\hat{b}_2} \end{bmatrix}$$

and thus

$$\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D} = (s^2_{\hat{b}_1}\hat{a}_2^2 + s^2_{\hat{a}_1}\hat{b}_2^2)(z_1-z_2)^2 + 2(\hat{a}_1\hat{a}_2s_{\hat{b}_1\hat{b}_2} + \hat{b}_1\hat{b}_2s_{\hat{a}_1\hat{a}_2})(z_1-z_2)^2$$
$$+ 2(\hat{a}_2^2s_{\hat{b}_1\hat{b}_2} + \hat{b}_2^2s_{\hat{a}_1\hat{a}_2})(z_1-z_2)(z_1^2-z_2^2)$$
$$+ s^2_{\hat{b}_2}(\hat{a}_1(z_1-z_2) + \hat{a}_2(z_1^2-z_2^2))^2 + s^2_{\hat{a}_2}(\hat{b}_1(z_1-z_2) + \hat{b}_2(z_1^2-z_2^2))^2$$

$$\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\} = s^2_{\hat{a}_1}s^2_{\hat{b}_2}(z_1-z_2)^2 + s^2_{\hat{a}_2}s^2_{\hat{b}_1}(z_1-z_2)^2 + s^2_{\hat{a}_2}s^2_{\hat{b}_2}(z_1^2-z_2^2)^2$$
$$+ 2s_{\hat{a}_1\hat{a}_2}s_{\hat{b}_1\hat{b}_2}(z_1-z_2)^2 + 2(s_{\hat{a}_1\hat{a}_2}s^2_{\hat{b}_2} + s_{\hat{b}_1\hat{b}_2}s^2_{\hat{a}_2})(z_1^2-z_2^2)(z_1-z_2)$$

Therefore, the first-order test statistic is $Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}}}$ and the second-order test statistic is

$Z = \frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}}}$ accordingly.

Scenario 6: a is moderated by Z_1 and Z and b is moderated by Z_2 and Z

The null hypothesis H_0 is

$$a_1b_2(z_{21}-z_{22})+a_2b_1(z_{11}-z_{12})+a_2b_2(z_{11}z_{21}-z_{12}z_{22})+(a_1b_3+a_3b_1)(z_1-z_2) \\ +a_2b_3(z_{11}z_1-z_{12}z_2)+a_3b_2(z_1z_{21}-z_2z_{22})+a_3b_3(z_1^2-z_2^2)=0$$

where z_{11} and z_{12} are two values of the moderator variable Z_1, z_{21} and z_{22} are two corresponding values of the moderator variable Z_2, and z_1 and z_2 are two values of the moderator variable Z. The first-order and second-order variance estimates of the point estimate $f(\hat{\boldsymbol{\theta}})$ can be derived by $\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}$ and $\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}$. Here, we have

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1\,\hat{a}_2\,\hat{a}_3\,\hat{b}_1\,\hat{b}_2\,\hat{b}_3]$$

$$f(\hat{\boldsymbol{\theta}}) = \hat{a}_1\hat{b}_2(z_{21}-z_{22})+\hat{a}_2\hat{b}_1(z_{11}-z_{12})+\hat{a}_2\hat{b}_2(z_{11}z_{21}-z_{12}z_{22})+(\hat{a}_1\hat{b}_3+\hat{a}_3\hat{b}_1)(z_1-z_2) \\ +\hat{a}_2\hat{b}_3(z_{11}z_1-z_{12}z_2)+\hat{a}_3\hat{b}_2(z_1z_{21}-z_2z_{22})+\hat{a}_3\hat{b}_3(z_1^2-z_2^2)$$

$$\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s^2_{\hat{a}_1} & s_{\hat{a}_1\hat{a}_2} & s_{\hat{a}_1\hat{a}_3} & 0 & 0 & 0 \\ s_{\hat{a}_1\hat{a}_2} & s^2_{\hat{a}_2} & s_{\hat{a}_2\hat{a}_3} & 0 & 0 & 0 \\ s_{\hat{a}_1\hat{a}_3} & s_{\hat{a}_2\hat{a}_3} & s^2_{\hat{a}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2_{\hat{b}_1} & s_{\hat{b}_1\hat{b}_2} & s_{\hat{b}_1\hat{b}_3} \\ 0 & 0 & 0 & s_{\hat{b}_1\hat{b}_2} & s^2_{\hat{b}_2} & s_{\hat{b}_2\hat{b}_3} \\ 0 & 0 & 0 & s_{\hat{b}_1\hat{b}_3} & s_{\hat{b}_2\hat{b}_3} & s^2_{\hat{b}_3} \end{bmatrix}$$

$$\boldsymbol{D}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} \hat{b}_2(z_{21}-z_{22})+\hat{b}_3(z_1-z_2) \\ \hat{b}_1(z_{11}-z_{12})+\hat{b}_2(z_{11}z_{21}-z_{12}z_{22})+\hat{b}_3(z_{11}z_1-z_{12}z_2) \\ \hat{b}_1(z_1-z_2)+\hat{b}_2(z_1z_{21}-z_2z_{22})+\hat{b}_3(z_1^2-z_2^2) \\ \hat{a}_2(z_{11}-z_{12})+\hat{a}_3(z_1-z_2) \\ \hat{a}_1(z_{21}-z_{22})+\hat{a}_2(z_{11}z_{21}-z_{12}z_{22})+\hat{a}_3(z_1z_{21}-z_2z_{22}) \\ \hat{a}_1(z_1-z_2)+\hat{a}_2(z_{11}z_1-z_{12}z_2)+\hat{a}_3(z_1^2-z_2^2) \end{bmatrix}$$

and

$$\boldsymbol{H}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & z_{21}-z_{22} & z_1-z_2 \\ 0 & 0 & 0 & z_{11}-z_{12} & z_{11}z_{21}-z_{12}z_{22} & z_1z_{11}-z_{12}z_2 \\ 0 & 0 & 0 & z_1-z_2 & z_1z_{21}-z_2z_{22} & z_1^2-z_2^2 \\ 0 & z_{11}-z_{12} & z_1-z_2 & 0 & 0 & 0 \\ z_{21}-z_{22} & z_{11}z_{21}-z_{12}z_{22} & z_1z_{21}-z_2z_{22} & 0 & 0 & 0 \\ z_1-z_2 & z_1z_{11}-z_2z_{12} & z_1^2-z_2^2 & 0 & 0 & 0 \end{bmatrix}$$

Since the final forms of $\mathrm{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}$ and $\mathrm{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}$ are very cumbersome, we do not display them here for the sake of saving space. The first-order test statistic is $Z=\frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathrm{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}}}$ and the second-order test statistic is $Z=\frac{f(\hat{\boldsymbol{\theta}})}{\sqrt{\mathrm{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\boldsymbol{D}+\frac{1}{2}tr\{(\boldsymbol{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}}}$ accordingly.

References

Preacher, K. J., Rucker, D. D., & Hayes, A. F. (2007). Addressing moderated mediation hypotheses: Theory, methods, and prescriptions. *Multivariate Behavioral Research*, *42 (1)*, 185 – 227.