Name ______________________________ Date ___________________

Math 10250 Activity 9: Compound Interest and the Number e (Section 2.2)

GOAL: Understand compounding in continuous time as the limiting case of n-times per year compounding.

Last time: Let $A(t)$ be the balance at time t (in years) of a bank account earning interest at an annual rate r (in decimals) compounded n times a year. Then we have:

$$\boxed{A(t) = P\left(1+\frac{r}{n}\right)^{tn}, \text{where } P \text{ is the principal; i.e. } A(0) = P}$$

Example 1 The balance $M(t)$ of a retirement account with interest compounded daily is given by the formula $M(t) = 30000(1.00022)^{365t}$. What is the principal and the annual interest rate?

(Ans: $P = \$30000$; $r = 8\%$)

Next, we want to consider the balance of an account where interest is compounded continuously; i.e., we are earning interest every instant the money is with the bank. (Good deal?)

▶ The number e

In the general formula above, if $P = 1, r = 1$ and $t = 1$ then $A(1) = \left(1+\frac{1}{n}\right)^n$.

Letting n go to ∞ we obtain that:

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \stackrel{?}{=} \qquad .}$$ ← balance at end of 1 yr. of an investment of \$1 at an annual interest rate of 100% compounded continuously

Example 2 Estimate e by completing the table:

n	1	2	10	100	1000
$\left(1+\frac{1}{n}\right)^n$					

Continuously compounded interest

Compute the limit:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \qquad\qquad = \qquad\qquad = \qquad\qquad .$$

↑ letting $m = n/r$, so that $n = mr$ ↑ by definition of e

Setting: As above except now $n \to \infty$.

The amount after t years with **continuously compounded interest** is:

$$A(t) = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{tn} = \qquad\qquad = P\cdot\lim_{n\to\infty}\left[\qquad\qquad\right]^t = \qquad\qquad .$$

<u>General formula</u>: $\boxed{A(t) = Pe^{rt}}$ ←(rate)(time in years)

↑ amount in account at end of t years ↖ initial amount invested (principal)

Example 3 If you open an account paying 9% interest, compounded continuously, then how much should you deposit to insure that there will be $60,000 in 15 years? Ans. $60,000e^{-1.35}$

Example 4 $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{3n} \stackrel{?}{=}$ Ans. $e^{3/2}$

Example 5 Suppose you put $5000 in an account paying 4% annual interest, and you leave it there without adding or withdrawing anything. How much will you have at the end of 3 years if the interest is compounded:

(a) 6 times a year? Ans. $\$5,635.24$

(b) 24 times a year? Ans. $\$5,636.92$

(c) continuously? Ans. $\$5,637.48$

Remark: What could you conclude from the answers obtained in Example 5?

▶ The natural exponential function

Recall: The exponential function is $f(x) = b^x$, where b is a positive constant. The most **popular** b is e.

Definition: The **natural exponential function** is $f(x) = e^x$.

Example 6 Graph the natural exponential function and its inverse. Write down all intercepts and asymptotes of the natural exponential function. Also, recall the laws of exponents with basis $b = e$.