Name ________________________________ Date ____________________

Math 10250 Activity 11: Natural Logarithm and Applications (Section 2.4)

GOAL: Define the **natural** logarithmic function $\ln x$ as the inverse of the **natural** exponential function, $f(x) = e^x$ and use it to solve equations when the unknown is an exponent as is the case when we need to determine doubling time or half-life time.

Last time: We met the logarithmic function with base b. Recall, $\boxed{\log_b x = y \Leftrightarrow \qquad\qquad, \quad x > 0}$.

Q1: What do we get when we let $\boldsymbol{b = e}$?

A1: The **natural logarithm**, $\ln x = \log_e x$, $x > 0$. Therefore $\boxed{\ln x = y \quad \Leftrightarrow \qquad\qquad, \quad x > 0.}$

• Since $\boldsymbol{\ln x}$ is the **inverse** of $\boldsymbol{e^x}$, we have the following two useful formulas:

$$\boxed{\ln(e^x) = \qquad, \quad \text{any } x} \quad \text{and} \quad \boxed{e^{\ln x} = \qquad, \quad x > 0.}$$

Sketch the graph of $\ln x$:

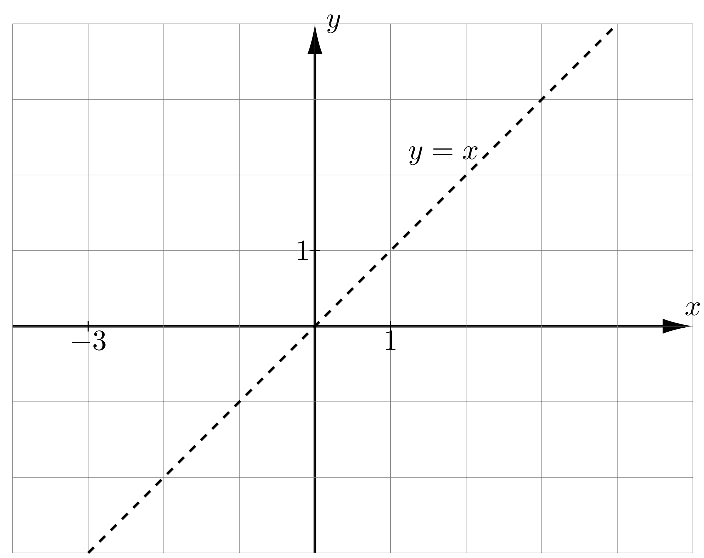
Q2: What are the **basic properties of $\ln x$**?

A2: • domain $\stackrel{?}{=}$ and range $\stackrel{?}{=}$

• It's continuous and increasing.

• $\lim\limits_{x\to\infty} \ln x \stackrel{?}{=}$ and $\lim\limits_{x\to 0^+} \ln x \stackrel{?}{=}$.

• $\ln 1 \stackrel{?}{=}$, $\ln e \stackrel{?}{=}$, and $\ln(1/e) \stackrel{?}{=}$



Example 1 Sketch the graph of $y = \ln(3 + x)$.

Example 2 Solve $e^{3-2x} = 8$ for x.

▶ Converting exponentials from base b to base e

Q3: How do we convert b^x to $e^{(\text{something})}$?

A3: Using $\boldsymbol{b = e^{\ln b}}$ we have the **conversion formula**: $\boxed{b^x = (\qquad\quad)^x = \qquad\qquad .}$

Example 3 Rewrite $\sqrt[3]{7}$ as an exponential with base e.

Example 4 Evaluate the given expression as a number in decimal form without using a calculator.

(a) $\ln\left(\dfrac{1}{\sqrt[4]{e}}\right)$

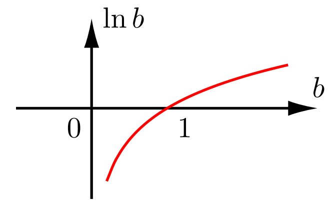
(b) $e^{2\ln 3}$

Example 5 Simplify $e^{\ln(5x)+\ln(2/x)}$.

▶ Exponential growth and decay

Recall: In Section 2.1 we saw that the equation for exponential growth and decay is:

$$\boxed{y = y_0 b^t = y_0 e^{(\ln b)t}},$$



since $b^x = e^{(\ln b)x}$.

- If $b > 1$ then $\ln b$ = growth constant. $\leftarrow$ exponential growth
- If $0 < b < 1$ then $\ln b < 0$. $|\ln b|$ = decay constant. $\leftarrow$ exponential decay

Example 6 If \$10,000 is deposited in an account paying 5% interest per year, compounded continuously, how long will it take for the balance to reach \$20,000?

Example 7 Polonium-210 has a decay constant of 0.004951, with time measured in days. How long does it take a given quantity of polonium-210 to decay to half the initial amount? In other words, what is the half-life of polonium-210?

Fact: For any radioactive substance: Half-life = -

Example 8 A bacteria culture starts with 500 bacteria and is growing exponentially. After 3 hours there are 8000 bacteria.

(a) Find a formula of the form $y = Ae^{kt}$ for the number of bacteria after t hours.

(b) Find the number of bacteria after 4 hours.

(c) When will the population reach 30,000?

Application (Log-Normal Model) In Finance and Economics a theoretical model for the value of the stock market $S(t)$ is given by the formula

$$\boxed{S(t) = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t} e^{\sigma\sqrt{t}\,Z},}$$

where Z is a standard normal random variable, r is the risk free interest rate, σ is the volatility, and S_0 is the value of the stock market at time $t = 0$. Take the natural logarithm of this formula and see if you can understand it better.