Name ________________________________ Date ____________________

Math 10250 Activity 16: Differentiability and Linear Approximation (Section 3.4)

GOAL: To approximate a differentiable function near a given point $x = a$ with the equation of its tangent line (a simpler linear function) at $x = a$. Discuss continuity versus differentiablity.

▶ **Differentiability** A function $f(x)$ is said to be differentiable if each point of its graph has a non-vertical tangent line. This means that the slope at each point of the graph is a ____________ number.

Graphically, differentiable means that each small segment of the graph of $f(x)$ is almost identical to a straight line. This is illustrated in Figures 1 through 3 below. As you zoom into the point $(a, f(a))$, the segment of the graph of $f(x)$ near point a becomes more and more like its tangent line at $x = a$.

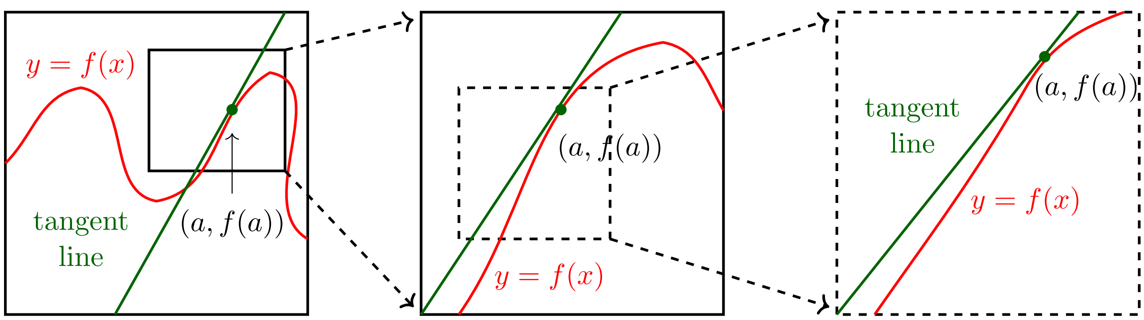


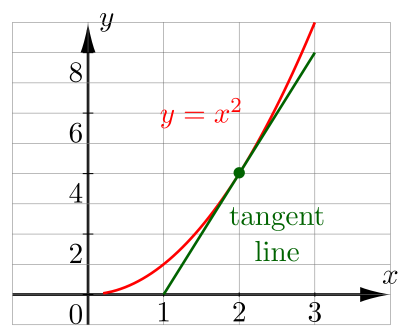
Figure 1. Figure 2. Figure 3.

Q1: Referring to Figure 3, what is the equation of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$?

Since the graph of $f(x)$ near point a is almost the same as its tangent line at $x = a$, we have:

$f(x) \approx$.	← Linear approximation of $f(x)$ near point a

Remark: If $f(x)$ is a differentiable function at $x = a$, the **two** values $f(a)$ and $f'(a)$ allow us to **estimate** the value of $f(x)$ when x is close to a!



Example 1 (a) Find the tangent line to $f(x) = x^2$ at $x = 2$.

(b) Give the tangent line approximation of $f(x)$ near 2.

(c) Using your answer in (b), estimate the following values and comment on their accuracy:

(i) $f(2.01) \overset{?}{\approx}$ (ii) $f(1.9) \overset{?}{\approx}$ (iii) $f(3) \overset{?}{\approx}$

Example 2 Apply linear approximation to the function $f(x) = x^{1/2}$ to estimate $\sqrt{25.5}$.

Example 3 Use the tangent line approximation to estimate $\sqrt[3]{26.31}$.

Example 4 The cost of producing 200 units of a certain item is \$5,000, and the marginal cost of producing 200 units is \$100. Use linear approximation to estimate the cost of producing 202 units.

Example 5 Assume that a population grows according to the (exponential) model $\frac{dP}{dt} = 0.02P$. If the population now is 5 millions, use linear approximation to estimate this population 10 years later. (Ans: 6 millions)

▶ Differentiability and continuity

A **continuous** function is **NOT** differentiable if the graph has a **corner** or a **vertical** tangent line

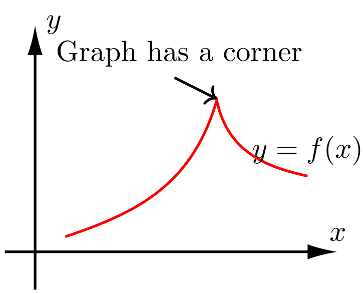


Figure 4

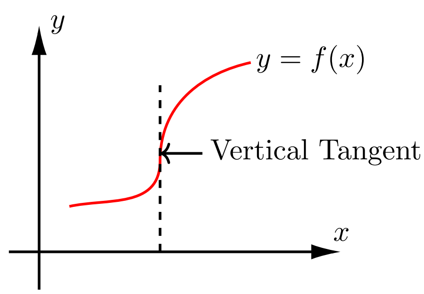


Figure 5

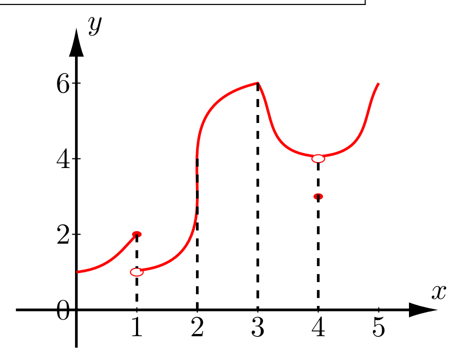


Figure 6

Example 6 According to figure 6, a) $f(x)$ is discontinuous at $x \stackrel{?}{=}$

b) $f(x)$ is not differentiable at $x \stackrel{?}{=}$

Definition: A function $f(x)$ is differentiable at point $x = a$ if the graph of $f(x)$ has a non-vertical tangent line at $(a, f(a))$. In terms of slope and limits, this means that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \textbf{exists} \text{ and is a } \textbf{finite} \text{ number}$$

Example 7 (a) Sketch the graph of $f(x) = |x|$ and decide by visual inspection whether $f(x)$ is differentiable at $x = 0$.

(b) Now, use the limit definition to decide whether $f(x)$ is differentiable at $x = 0$.

Remark: $f(x) = |x|$ is a function that is continuous but NOT differentiable.

Q2: Can a differentiable function not be continuous? **A2:**_____________

Theorem. *If a function f is differentiable at a, then it is continuous at a.*