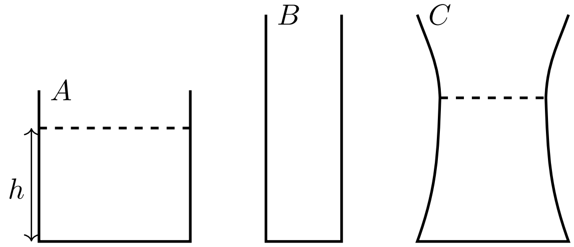
Name ______________________________ Date ___________________

Math 10250 Activity 23: Second Derivative Tests (Section 4.2)

GOAL: To study how the graph of a given $f(x)$ "bends", and how these features of the graph are described by $f'(x)$ and $f''(x)$.

▶ The second derivative test for concavity

Example 1 Water is filling up each of the following cylindar vessels at a constant rate of 1 cm^3/sec.



Let h be the height of the water level in the vessel at time t.

a. Sketch the graphs of h versus t for Vessels A and B in the axes for Figure 1. Indicate which graph belongs to A and which to B.

Figure 1

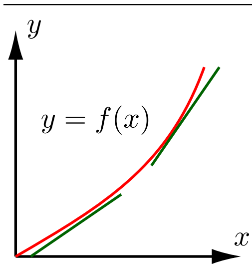
Figure 2

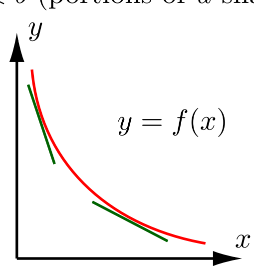
b. Sketch the graph of h versus time t for Vessel C in the axes for Figure 2.

c. Comment on how the "bending" (up or down) of the graph changes with $h'(t)$. Mark on the graph where the "bending" changes.

We now introduce terminologies that describe the "bending" of a graph.

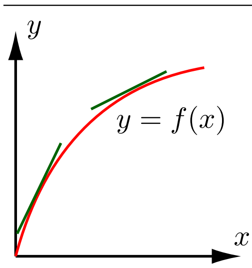
Case 1: For $a < x < b$, the slope of the graph $f(x)$ is **increasing** as x increases (i.e., $f'(x)$ is increasing). So $f''(x)$ is ____________________ for $a < x < b$ (portions of u-shape).

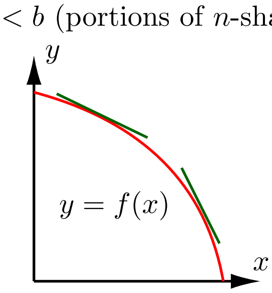




We say that the graph of $f(x)$ is ________________ for $a < x < b$.

Case 2: For $a < x < b$, the slope of the graph $f(x)$ is **decreasing** as x increases (i.e., $f'(x)$ is decreasing). So $f''(x)$ is ____________________ for $a < x < b$ (portions of n-shape).





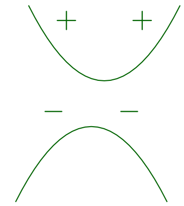
We say that the graph of $f(x)$ is ________________ for $a < x < b$.

The above gives us:

The Second derivative test for concavity

Let $f(x)$ be a function that has a second derivative in an interval.

- If $\boldsymbol{f''(x) > 0}$ for all x then its graph is ____________________.
- If $\boldsymbol{f''(x) < 0}$ for all x then its graph is ____________________.



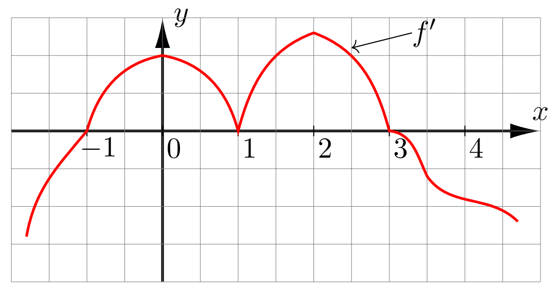
Note: The places where the graph of $f(x)$ changes its concavity are called ____________________.

Example 2 Using the graph of the derivative of $f(x)$ below, determine the concavity of $f(x)$.

Concave up:

Concave down:

Inflection points:



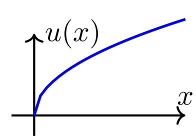
Q1: Where can $f''(x)$ change signs (i.e., $f(x)$ changes concavity)?

A1: At the points where (i) _____________, or (ii) _______ is undefined (e.g., f' has a sharp corner).

Example 3 The position of an object moving on a straight line is given by $s(t) = 2t^3 + 3t^2 - 36t + 7$. Determine (a) where the graph of $s(t)$ is concave up, (b) where it is concave down, and (c) where there are inflection points, if any. Give physical interpretations for each of (a), (b), and (c).

Example 4 Determine where the graph of $f(x) = x^{5/3}$ is concave up, where it is concave down, and where there are inflection points, if any. Sketch the graph of $f(x)$.

Application in Economics: Utility functions $u(x)$ are
- increasing $\iff u'(x) > 0$
- concave down $\iff u''(x) < 0$. (Like $u(x) = \sqrt{x}$)



Your turn (*Application to Population/Pandemics Model*): For the solution $y = y(t)$ of the logistic model below, show that its concavity changes when $y(t) = K/2$ (as the picture indicates).

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right), \quad y(0) = y_0.$$