Name ______________________________ Date ____________________

Math 10250 Activity 33: Area and the Definite Integral (Section 5.4)

Goal: To compute the area of **curved** regions in the plane and define the definite integral of "good" functions.

• Consider the region under the graph of a non-negative function $f(x)$ over its domain $[a, b]$:

Q1: How do you compute the area of the region A?

A1: In five steps:

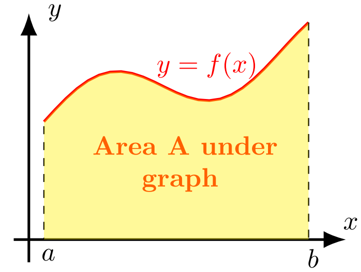
(1) Divide the interval $[a, b]$ into n equal subintervals.

(2) Choose a point in each subinterval.

(3) Compute the area of the rectangle corresponding to each piece.

(4) Estimate the area of A by adding the areas of all rectangles.

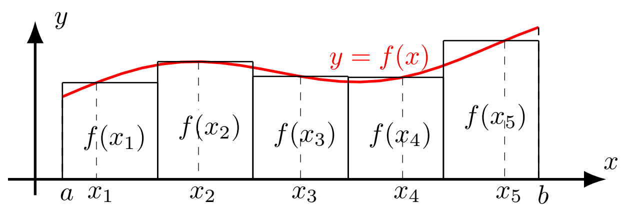
(5) Get the exact area by taking larger and larger n (smaller and smaller subintervals).



Let's look at each of the above steps in detail.

(1) Divide $[a, b]$ into n subintervals of equal width Δx and choose a point x_i in each subinterval.

(Usually this point is chosen to be either the left endpoint, the right endpoint, or the midpoint of the subinterval.)



$\Delta x =$ ______ ← **mesh** of the partition

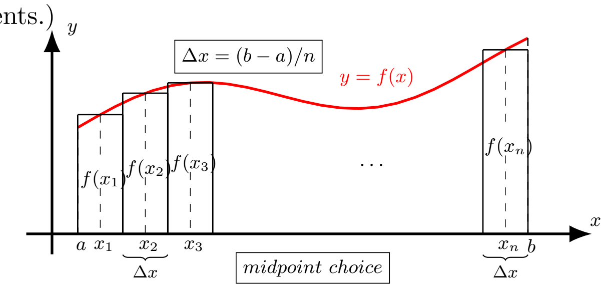
(2) Construct a rectangle over each subinterval with height $f(x_i)$ and compute the area of each rectangle. (Let's use left-hand endpoints of the segments.)

Area of first rectangle = height · base = ____________________.

Area of second rectangle = height · base = ____________________.

⋮

Area of nth rectangle = height · base = ____________________.



(3) Estimate the area of A by adding the areas in (2).

area of $A \approx$ ______________________ $= S_n(f)$ ← **Riemann Sum**

(4) The approximation above gets more accurate as the rectangles get smaller.

(5) So we can get the exact area of A by letting ____________. Therefore,

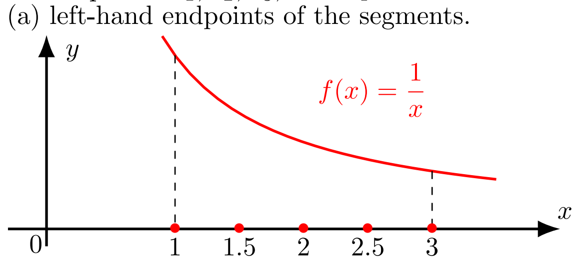
$$\text{area of } A = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} S_n(f)$$

Example 1 Estimate the area under the graph of $y = 1/x$, $1 \le x \le 3$, by partitioning the interval $[1, 3]$ into 4 equal segments and computing the Riemann sum

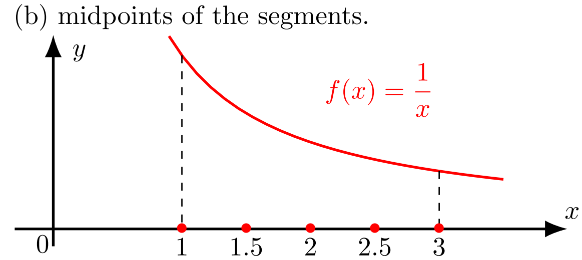
$$S_4(f) = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x,$$

where the points x_1, x_2, x_3, and x_4 are chosen to be:

(a) left-hand endpoints of the segments.



(b) midpoints of the segments.



▸ The definite integral: nonnegative case

The limit in Step (5) on the previous page is so special that we give it a name and symbol:

$$\boxed{\qquad\qquad = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]}$$ ← **Definition of the Definite Integral**

↑ area under the graph of $f(x)$ for $a \le x \le b$ if $f(x)$ is nonnegative

If this limit exists, we call it the **Definite Integral of** $f(x)$ **over the interval** $a \le x \le b$.

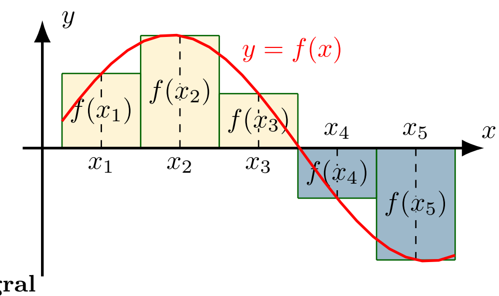
Example 2 Find $\int_0^1 4x\ dx$ using geometry.

▸ The Definite integral of a function taking positive and negative values.

We do it In four steps:

(1) Divide the interval $[a, b]$ into n subintervals.

(2) Choose a point in each subinterval.



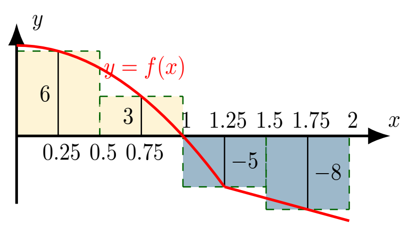
(3) Compute the corresponding Riemann sum.

$S_n(f) =$ ← **Riemann Sum**

(4) Let n go to infinity and obtain:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right].$$ ← **Definite Integral**

Example 3 For the function $f(x)$ whose graph is displayed in the figure on the right, estimate $\int_0^2 f(x)dx$ by using the Riemann sum corresponding to $\Delta x = 0.5$ and the midpoints.

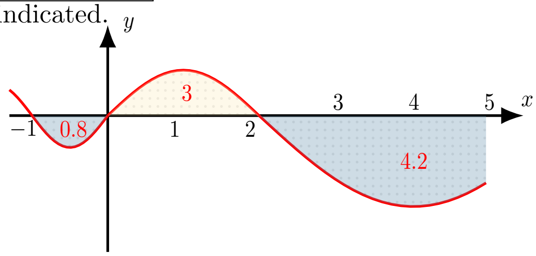


Example 4 Estimate the integral $\int_{-1}^{1} x^3 e^{-x^2} dx$ using 4 subintervals and left-hand endpoints.

• The relation between integral and area is:

$$\int_a^b f(x)\ dx = (\text{area of region lying \_\_\_\_\_\_\_\_\_\_ the } x\text{-axis}) - (\text{area of region lying \_\_\_\_\_\_\_\_\_\_ the } x\text{-axis}).$$

Example 5 The graph of $f(x)$ for $-1 \le x \le 5$ is shown in the figure below. The size of each enclosed area is as indicated.



$\int_{-1}^{0} f(x)\ dx \stackrel{?}{=}$ $\qquad$ $\int_{0}^{2} f(x)\ dx \stackrel{?}{=}$ $\qquad$ and $\int_{2}^{5} f(x)\ dx \stackrel{?}{=}$

(a) Find the area of the region **enclosed** by the graph of $f(x)$, $-1 \le x \le 5$, and the x-axis.

(b) $\int_{-1}^{5} f(x)dx \stackrel{?}{=}$

Q2: What are the basic properties of definite integral?
A2: