Name _____ Date _____

Math 10250 Activity 36: Continuous Income Streams (Section 6.2)

Goal: To compute Future Value (FV) and Present Value (PV) of a continuous income streams, which like water it flows into an account earning interest continuously.

•Recall that we calculated the future value (FV) assuming that we made only *one* initial deposit and no other deposits or withdrawals.

Example 1 Jen plans to retire in 30 years and has arranged for money to be continuously deposited into an account at a rate of \$10,000 per year.

Q1: What is the FV of the income in 30 years if the interest is compounded continuously at a rate of 7%?

A1: The idea is that the FV is, in fact, an integral and we know that **integral ≈ Riemann sum**.
Q2: What is the Riemann sum for Jen? Start with partitioning [0, 30] into n (say $n = 3 \cdot 10^{10}$) subintervals of length $\Delta t = 30/n$ (say $\Delta t = 1/10^9$).

A2: Consider the following table.

Interval	$[0, t_1]$	$[t_1, t_2]$	$[t_2, t_3]$	$\ldots$	$[t_{n-1}, 30]$
Amt. flowing into account in interval	$10,000 \cdot \Delta t$				
FV of amt. at end of 30 years	$10,000 \cdot \Delta t \cdot e^{0.07(30-t_1)}$				

• Adding up all FVs above gives us

$$\text{Total } FV \approx 10,000 \cdot e^{0.07(30-t_1)} \cdot \Delta t + 10,000 \cdot e^{0.07(30-t_2)} \cdot \Delta t + \cdots + 10,000 \cdot e^{0.07(30-t_n)} \cdot \Delta t.$$

Q3: What is the integral for the FV of Jen's account?

A3: Letting $n \to \infty$ gives us

$$\boxed{\text{Future Value of income stream} = \int \quad .}$$

Q4: Now what if Jen plans to retire in T years and deposits at a rate of S dollars per year with interest rate r?

A4: $\boxed{\text{FV} = \int \quad .}$

Q5: Even better, what if Jen deposits at a rate of $S(t)$ dollars per year? Note that here the rate of deposit is not constant; it changes over time.

A5: The FV of a continuous income stream flowing at a rate of $S(t)$ dollars per year for T years, earning interest at an annual rate r, compounded continuously, is given by

$$\boxed{\boxed{\text{FV} = \int \quad .}}$$

Q6: What is the **PV** of the same income stream in A4?

A6:

$$\boxed{\text{PV} = \int \quad .}$$

Note: This PV is the amount of money you would have to invest in an account to have FV at the end of T years (that is compounded continuously with interest rate r).

Example 2 Suppose that money is deposited steadily into a savings account at a constant rate of \$30,000 per year. Find the balance at the end of 5 years if the account pays 10% interest, compounded continuously.

Example 3 Your company offers you the following two options:

(a) For the next 10 years it deposits money continuously into an account A at a rate of $2000e^{0.1t}$ dollars per year.

(b) At the beginning it deposits \$25,000 into an account B and nothing more during the next 10 years.

If both accounts yield 5% interest, compounded continuously, which option will you choose? Explain your answer. (Hint: Compute PV of the income stream in (a).)

Ans: PV(a) = \$25,948

Example 4 (a) In 2018, the Social Security program was providing monthly benefits to about 62 million people and total benefit payments for the year were about \$0.94 trillion[1]. Assume that for the next 10 years Social Security continues to pay benefits at this rate (\$0.94 trillion) per year steadily.

(b) On the other hand, in 2018, the Social Security had a surplus (Trust-fund) of about \$2.85 trillion, and an income stream of about \$0.98 trillion per year. Assume, again, that for the next 10 years the Social Security continues to receive this income at the same rate (\$0.98 trillion) per year steadily.

Find the Social Security surplus after 10 years, if during this period the prevailing annual interest rate is 3% compounded continuously. (Hint: Compute the difference of the two FV's.)

Ans: Surplus = \$4.31 trillion

[1] *Source*: 2017 and 2018 Trustees Reports.