Math 10250 Activity 4: Limits (Sect. 1.1)

GOAL: To obtain an intuitive understanding of the fundamental concept of limit and learn rules for computing it.

Q1: Using your intuition, how would you interpret the statement: The function \( f(x) = \frac{x^2 - 2x - 3}{x - 3} \) has limit 4 as \( x \) goes to 3?

\( \frac{x^2 - 2x - 3}{x - 3} \approx 4 \) when \( x \) is close to 3, but \( x \neq 3 \).

A1: -Natural domain of \( f \): \( x \neq 3 \).
   -Since \( f \) is not defined at \( x = 3 \), let's look at how \( f \) behaves near \( x = 3 \). To do this, we make a table of values like this:

<table>
<thead>
<tr>
<th>( x )</th>
<th>2.97</th>
<th>2.98</th>
<th>2.99</th>
<th>3</th>
<th>3.01</th>
<th>3.02</th>
<th>3.03</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{x^2 - 2x - 3}{x - 3} )</td>
<td>3.97</td>
<td>3.98</td>
<td>3.99</td>
<td>?</td>
<td>4.01</td>
<td>4.02</td>
<td>4.03</td>
</tr>
</tbody>
</table>

Pattern: \( f(x) \) gets close to 4 as \( x \) gets close to 3.

- To make this more precise we need the help of algebra. So, let us factor the numerator of \( f \):

\( f(x) = \frac{x^2 - 2x - 3}{x - 3} = \frac{(x-3)(x+1)}{x - 3} \)

\( x \neq 3 \)
\( = x + 1 \)

- Letting \( x \to 3 \) gives \( f(x) = x + 1 \to 3 + 1 = 4 \)

i.e.
- \( f(x) \approx 4 \) for all \( x \) near 3 (but \( x \neq 3 \)), and
- Can make \( f(x) \) as close to \( L \) as we wish by taking \( x \) close enough to 3.

-Now, we are confident to claim that the limit of \( f(x) \) as \( x \) goes to 3 is 4.

- We write this as: \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = 4 \).

Q2: Give an Informal Definition of Limit

A2: \( \lim_{x \to a} f(x) = L \) if:

- \( f(x) \approx L \) if \( x \) is close to \( a \) (but not equal to \( a \))

- Can make \( f(x) \) as close to \( L \) as we wish by taking \( x \) close enough to \( a \).
**Exercise 1** The graph of a function \( f \) is shown in Figure 2. By inspecting the graph, find each of the following limits if it exists. If the limit does not exist, explain why.

(i) \( \lim_{x \to 4} f(x) = 0 \)

(ii) \( \lim_{x \to 1} f(x) = 1 \)

(iii) \( \lim_{x \to 2} f(x) = 4 \)

(iv) \( \lim_{x \to 0} f(x) \) does not exist

(v) \( \lim_{x \to 3} f(x) \) does not exist

![Figure 2](image)

**Exercise 2** Find \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \). Complete the following table of values to guess the limit and then use algebra to justify it (as in A).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x^2 - 4}{x - 2} )</td>
<td>?</td>
<td>3.999</td>
<td>?</td>
<td>4.001</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

\[
\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2
\]

\[
\cdot x + 2 \xrightarrow{x \to 2} 2 + 2 = 4
\]

Q3: What are the basic Limit Laws?

A3:

1. \( \lim_{x \to a} c f(x) = c \lim_{x \to a} f(x) \)

2. \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \)

3. \( \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \)

4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) \( \text{if} \lim_{x \to a} g(x) \neq 0 \)

5. \( \lim_{x \to a} [f(x)]^r = [\lim_{x \to a} f(x)]^r \)

**Exercise 3** Determine the following limits using the properties of limits (i.e. limit laws) and simplifying the expression, if necessary.

(i) \( \lim_{x \to 5} x^4 = 5^4 \)

(ii) \( \lim_{x \to 2} (5x^3 + 4x^2) = 5 \lim_{x \to 2} x^3 + 4 \lim_{x \to 2} x^2 = 5 \cdot 2^3 + 4 \cdot 2^2 \)

(iii) \( \lim_{x \to 2} (5x^3 + 4x^2) \cdot (x^2 - 9) = \left[ \lim_{x \to 2} (5x^3 + 4x^2) \right] \cdot \left[ \lim_{x \to 2} (x^2 - 9) \right] = \ldots \)

(iv) \( \lim_{x \to 2} \frac{x^2 - 9}{x - 3} = \lim_{x \to 2} \frac{x^2 - 9}{x - 3} = \frac{2^2 - 9}{2 - 3} = \frac{-5}{-1} = 5 \)

(v) \( \lim_{h \to 0} \frac{(h - 2)^2 - 4}{h} = \lim_{h \to 0} \frac{h^2 - 4h + 4 - 4}{h} = \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h - 4)}{h} = \lim_{h \to 0} (h - 4) = -4 \)

**Exercise 4** If \( f(x) \) is the function of exercise 1 and \( g(x) = 3x + 2 \) then find the following limits:

(i) \( \lim_{x \to 2} [f(x) \cdot g(x)] = [\lim_{x \to 2} f(x)] \cdot [\lim_{x \to 2} g(x)] = 4 \cdot [3 \cdot 2 + 2] = 32 \)

(ii) \( \lim_{x \to 2} \sqrt{f(x)} = \sqrt{\lim_{x \to 2} f(x)} = \sqrt{4} = 2 \).

An. 32

An. 2

2