Math 10250 Activity 9: Compound Interest and the Number $e$ (Sec. 2.2)

**Last time:** Let $A(t)$ be the balance at time $t$ (years) of a bank account earning interest at an annual rate $r$ (in decimals) compounded $n$ times a year. Then we have:

$$A(t) = P \left(1 + \frac{r}{n}\right)^{tn}$$

where $P$ is the principal i.e. $A(0) = P$.

**Example 1** The balance $M(t)$ of a retirement account with interest compounded daily is given by the formula $M(t) = 30000(1.00022)^{365t}$. What is the principal and the annual interest rate?

$$30000 \cdot 365 \log(1.00022) = P (1.00022)^{365t}$$

$$\Rightarrow \quad M(t) = P (1.00022)^{365t}$$

$$\Rightarrow \quad P = 30000$$

$$\Rightarrow \quad \log(1.00022) = \frac{r}{365}$$

$$\Rightarrow \quad r = 0.08$$

Next, we want to consider the balance of an account where interest is compounded continuously i.e. we are earning interest every instant the money is with the bank. (Good deal?)

▶ The number $e$

In the general formula above, if $P = 1, r = 1$ and $t = 1$ then $A(1) = \left(1 + \frac{1}{n}\right)^n$.

Letting $n$ go to $\infty$ we obtain that:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$\leftarrow$ balance at end of 1 yr. of an investment of $\$1$ at an annual interest rate of 100% compounded continuously

**Example 2** Estimate $e$ by completing the table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 + \frac{1}{n})^n$</td>
<td>2.00</td>
<td>2.25</td>
<td>2.59</td>
<td>2.70</td>
<td>2.716</td>
</tr>
</tbody>
</table>

Continuously compounded interest

Compute the limit:

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{mn} = \left(\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m\right)^r = e^r$$

Letting $m = n/r$, so that $n = mr$

by definition of $e$

Setting: As above except now $n \to \infty$

The amount after $t$ years with **continuously compounded interest** is:

$$A(t) = \lim_{n \to \infty} P \left(1 + \frac{r}{n}\right)^{tn} = P \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{tn} = P \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = P (e^r)^t = Pe^{rt}.$$

**General formula:**

$$A(t) = Pe^{rt} \quad \leftarrow \text{(rate)(time in years)}$$

amount in account at end of $t$ years

initial amount invested (principal)
Example 3 If you open an account paying 9% interest, compounded continuously, then how much should you deposit to insure that there will be $60,000 in 15 years?

\[ Pe^{0.09 \cdot 15} = 60,000 \quad \Rightarrow \quad P = \frac{60,000}{e^{1.35}} = 15554.42 \]

Example 4 \[ \lim_{n \to \infty} \left( 1 + \frac{1}{2n} \right)^{3n} = \lim_{m \to \infty} \left( 1 + \frac{1}{m} \right)^{m \cdot \frac{3}{2}} = e^{\frac{3}{2}} \]

Example 5 Suppose you put $5000 in an account paying 4% annual interest, and you leave it there without adding or withdrawing anything. How much will you have at the end of 3 years if the interest is compounded:

(a) 6 times a year? \[ A(t) = 5000 \left( 1 + \frac{0.04}{6} \right)^{3 \times 6} \]
\[ \approx 5635.24 \]

(b) 24 times a year? \[ A(t) = 5000 \left( 1 + \frac{0.04}{24} \right)^{3 \times 24} \]
\[ \approx 5636.92 \]

(c) continuously? \[ A(t) = 5000 e^{0.04 \cdot 3} = 5000 e^{0.12} \]
\[ \approx 5637.48 \]

Remark: What could you conclude from the answers obtained in Example 5? I will have the most money when the interest is compounded cont.

The natural exponential function

Recall: The exponential function is \( f(x) = b^x \), where \( b \) is a positive constant. The most popular \( b \) is \( e \).

Definition: The natural exponential function is \( f(x) = e^x \).

Example 6 Graph the natural exponential function and its inverse. Write down all intercepts and asymptotes of the natural exponential function.