GOAL: To learn how to compute the derivatives of a product and a quotient of two functions.

**The Product Rule:**  \[ \frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x) \]

Note: \[ \frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[g(x) \cdot f(x)]. \]

**Example 1** Use the product rule to find the derivatives:

(a) \[ \frac{d}{dx}[(2x^3 - x)^2] = (6x^2 - 1)(3x^2 - x) \]
\[ = 18x^4 - 6x^3 - 6x^2 + x \]

(b) \[ \frac{d}{dx}[e^{-2x} \ln x] = \frac{(-2e^{-2x})(\ln x) + e^{-2x}(\frac{1}{x})}{(-2x)(\ln x)} \]
\[ = -2e^{-2x}(\ln x) + e^{-2x}(\frac{1}{x}) \]

**The Quotient Rule:**  \[ \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \]

In general, \[ \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] \neq \frac{d}{dx} \left[ \frac{g(x)}{f(x)} \right]. \]

**Example 2** Find the equation of the tangent line to the graph of \[ y = \frac{x}{x^2 + 1} \] at the point \( x = 2 \).

\[ \frac{dy}{dx} = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \]
\[ f(2) = \frac{2}{5} \]
\[ f'(2) = \frac{1 - 2^2}{(2^2 + 1)^2} = -\frac{3}{25} \]

\[ y - \frac{2}{5} = -\frac{3}{25} (x - 2) \]

**Example 3** Use the appropriate differentiation rule you learned so far to find the derivatives below. Some algebra may be helpful.

(a) \[ \frac{d}{dx} \left( \frac{x^2 + x - 3}{100} \right) = \frac{(x^2 + x - 3)'(100) - (x^2 + x - 3)(100)'}{(100)^2} \]
\[ = \frac{(2x + 1)(100) - (x^2 + x - 3)(0)}{(100)^2} \]
\[ = \frac{100(2x + 1)}{(100)^2} \]
\[ = \frac{2x + 1}{100} \]

(b) \[ \frac{d}{dx} \left( \frac{x + e^x}{e^x} \right) = \frac{(x + e^x)'(e^x) - (x + e^x)(e^x)'}{(e^x)^2} \]
\[ = \frac{(1 + e^x)(e^x) - (x + e^x)(e^x)}{(e^x)^2} \]
\[ = \frac{1 - x}{e^x} \]

(c) \[ \frac{d}{dx} \left( \frac{\ln x}{x^2} \right) = \frac{(\ln x)'(x^2) - \ln x (x^2)'}{(x^2)^2} \]
\[ = \frac{\frac{1}{x} \cdot x^2 - \ln x (2x)}{x^4} \]
\[ = \frac{x - 2 \ln x}{x^4} \]
\[ = \frac{x(1 - 2 \ln x)}{x^4} \]
\[ = \frac{1 - 2 \ln x}{x} \]

(d) \[ \frac{d}{dx} \left( \frac{x^2 + x - 3}{x^{10}} \right) = \frac{(x^2 + x - 3)'x^{10} - (x^2 + x - 3)(x^{10})'}{(x^{10})^2} \]
\[ = \frac{2x + 1 \cdot x^{10} - (x^2 + x - 3) \cdot 10x^9}{x^{20}} \]
\[ = \frac{2x^{11} + 10x^{10} - 10x^{10} - 30x^9}{x^{20}} \]
\[ = \frac{-8x^{11} - 30x^9}{x^{20}} \]
Example 4 Suppose the demand for a certain product is given by \( q = f(p) \), where \( p \) is the price per unit and \( q \) is the number sold. The revenue is given by \( R = pq \).

(a) If \( f(300) = 20,000 \) and \( f'(300) = -30 \), find \( dR/dp \) when \( p = 300 \).

\[
R(p) = p \cdot q = p \cdot f(p) \\
\frac{dR}{dp} = f(p) + pf'(p) \\
\frac{dR(300)}{dp} = f(300) + 300f'(300) \\
= 20,000 + 300(-30) \\
= 20,000 - 9,000 = 11,000 > 0
\]

(b) If the product is currently selling for $300 per unit, should the company increase or decrease the price in order to raise the revenue?

Since \( R'(300) > 0 \), the company should increase the price.

Example 5 For what \( x \) does the graph \( y = xe^x \) have slope zero?

\[
0 = (xe^x)' = e^x + xe^x = e^x(1 + x) = 0 \implies x = -1
\]

Example 6 Find the equation of the tangent line to the graph of \( y = \frac{1 - \ln x}{1 + \ln x} \) at \( x = 1 \). Ans: \( y = -2x + 3 \)

\[
\frac{dy}{dx} = \frac{\left(1 - \ln x\right) - \left(1 - \ln x\right) \cdot \frac{1}{x}}{(1 + \ln x)^2} \\
= \frac{1}{x(1 + \ln x)} - \frac{1}{(1 + \ln x)^2} \\
= \frac{-2}{x(1 + \ln x)^2}
\]

\[
y(1) = \frac{1 - 0}{1 + 0} = 1 \\
y'(1) = \frac{-2}{1 + 1} = -1
\]

Example 7 Let \( p(x) = f(x)g(x) \) and \( q(x) = \frac{f(x)}{g(x)} \). Using the graph of \( f(x) \) and \( g(x) \) find

(a) \( p'(a) \)

\[
p'(x) = f'(x)g(x) + f(x)g'(x) \\
p'(a) = f'(a)g(a) + f(a)g'(a) \\
= 2 \cdot 10 + 5 \cdot (-2) = 10
\]

(b) \( q'(a) \)

\[
q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
q'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2} \\
= \frac{2 \cdot 10 - 5 \cdot (-2)}{10^2} = \frac{20 + 10}{100} = \frac{3}{10} \approx 0.3
\]

Ans: \( p'(a) = 10 \) and \( q'(a) = 0.4 \)