Math 10250 Activity 32: Integration by Parts and Partial Fraction Decomposition
(Section 5.3)

**GOAL:** To find integrals using Integration by Parts and Partial Fraction decomposition.

- **Integration by parts**

  **IDEA:** Recall that Integration by Substitution “reverses” chain rule. Today we learn another technique, called *integration by parts*, which “reverses” the product rule.

  - Let \( u(x) \) and \( v(x) \) be two differentiable functions. Applying product rule, we have:
    \[
    \frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + u'(x)v(x)
    \]
    
  - By definition of anti-derivative:
    \[
    u(x)v(x) = \int \left[ u(x)v'(x) + u'(x)v(x) \right] \, dx = \int u(x)v'(x) \, dx + \int u'(x)v(x) \, dx.
    \]
    
  - Rearranging terms, we have:
    \[
    \int u(x)v'(x) \, dx = u(x)v(x) - \int v(x)u'(x) \, dx
    \]
    
  - Note \( \frac{du}{dx} = u'(x) \Rightarrow \int u(x) \, dx = \frac{u'(x) \, dx}{x} \)
    Also \( \frac{dv}{dx} = v'(x) \Rightarrow \int v(x) \, dx = \frac{v'(x) \, dx}{x} \)

  - Suppressing variable \( x \), we get:
    \[
    \int u \, dv = uv - \int v \, du
    \]
    
    . → **Integration by Parts**

**Example 1** Use integration by parts to find the following integrals.

(a) \[
\int e^{3x} \, dx = \frac{1}{3} e^{3x} - \frac{1}{3} e^x + C
\]

(b) \[
\int x^3 \ln x \, dx = \frac{1}{4} x^4 \ln x - \frac{1}{4} x^4 + C
\]
Partial Fraction Decomposition

Example 2  Find \( \int \frac{2}{x^2 - 3x + 2} \, dx \) by first writing \( \frac{2}{x^2 - 3x + 2} = \frac{A}{x-1} + \frac{B}{x-2} \).

\[ \Rightarrow \frac{2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \]

\[ 2 = A(x-2) + B(x-1) \]

\[ \bullet \ x = 1 \Rightarrow 2 = A(-1) + B(1-1) \]

\[ 2 = -A \Rightarrow A = -2 \]

\[ \bullet \ x = 2 \Rightarrow 2 = A(2-2) + B(2-1) \]

\[ B = 2 \]

Example 3  Use any integration method to perform the following indefinite integrals:

(a) \( \int \frac{x}{\sqrt{2x+9}} \, dx = x \cdot \frac{\sqrt{2x+9}}{3} - \frac{\sqrt{2x+9}}{3} \, dx \)

\( u = x \Rightarrow du = dx \)

\[ dV = \sqrt{2x+9} \, dx \]

(b) \( \int \frac{x+1}{x^2 + 2x + 8} \, dx = \frac{1}{2} \ln |x| + c \)

\[ x^2 + 2x + 8 = u \]

\[ 2x \, dx + 2 \, dx = du \]

\[ 2 \, dx (x+1) = du \Rightarrow (x+1) \, dx = \frac{1}{2} \, du \]

(c) \( \int \frac{\ln x}{x} \, dx = \ln x \cdot x - \int x \, d(\ln x) \)

\[ = x \ln x - \frac{1}{2} x^2 \ln x + c \]

(d) \( \int \frac{5}{4 - x^2} \, dx = \frac{5}{4} \int \frac{1}{2-x} + \frac{5}{2+x} \, dx \)

\[ 5 = A(2+x) + B(2-x) \]

\[ \bullet \ x = 2 \Rightarrow 5 = A(4) + B \]

\[ 5 = 4A \Rightarrow A = \frac{5}{4} \]

\[ \bullet \ x = -2 \Rightarrow 5 = A(2-2) + B(2-2) \]

\[ 5 = -2B \Rightarrow B = \frac{5}{4} \]

Example 4  In a study of students learning foreign language, the number of new words \( w(t) \) (as a function of time) an average student can learn a day is modeled by the equation \( \frac{dw}{dt} = 0.1 (1-t) e^{-0.1t} \).

If the student begins with 20 new words a day, how many new words a day can he learn after 10 days?

\[ W(t) = (9 + t) e^{-0.1t} + 11 \]

\[ W(0) = (9 + 0) e^0 + 11 \]

\[ W(10) = (9 + 10) e^{-1} + 11 \]

\[ 20 = 9 + c \Rightarrow c = 11 \]