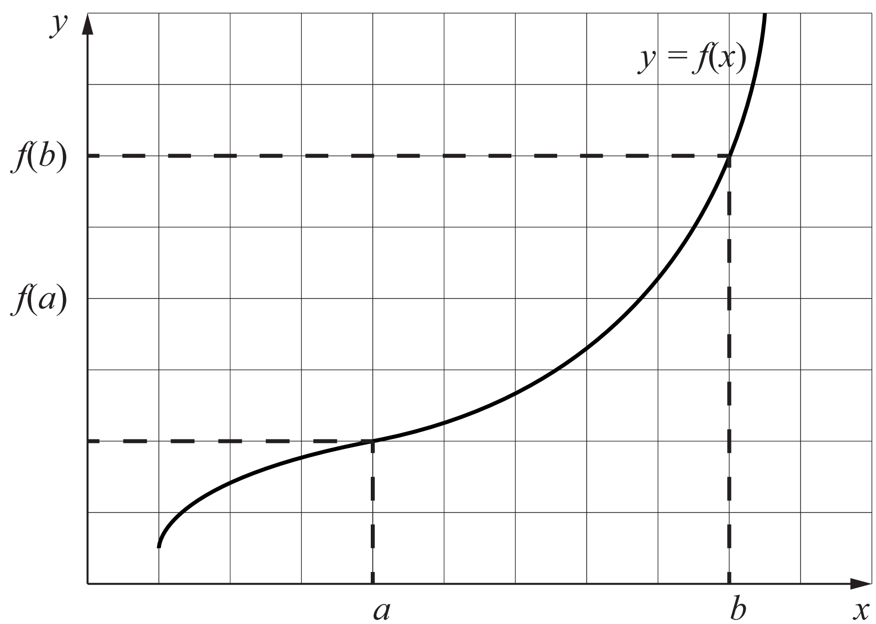
Math 10350 – Example Set 03A
Sections 2.1, 2.2, 2.3 & 2.4

The Average Rate of Change of a function $f(x)$ over the interval $[a, b]$

is given by $= \dfrac{\text{Change in } f(x)}{\text{Change in } x} = \dfrac{\Delta f}{\Delta x} =$ ______________________________ .

Sketch in the graph the chord (secant line) whose slope gives this average value.



In the special case when the function is the position $s(t)$ meter of a particle moving on a straight line at time t seconds, the average rate of change of the position over the time interval $a \leq t \leq b$ is also called the

____________________________________ over the time interval $a \leq t \leq b$

$= \dfrac{\text{Change in position}}{\text{Change in time}} = \dfrac{\Delta s}{\Delta t} =$ ________________________ m/sec.

The Average Velocity and Instantaneous Velocity

1. The position (vertical height measured from the ground) of a ball projected vertically up from the ground is given by $s(t) = 30t - 5t^2$ meter at time t second. Find each of the following values and simplifying your answer.

(1a) Average rate of change of the position of the ball over the time interval $1 \leq t \leq 4 =$

Average velocity of the ball over time interval $1 \leq t \leq 4 = \frac{\text{Change in position}}{\text{Change in time}} =$

(1b) Average velocity over the time duration between 1 and t (assuming $t \neq 1$) $=$

(1c) Complete the table:

t	0.99	0.999	0.9999	1	1.0001	1.001	1.01
$\frac{s(t)-s(1)}{t-1}$				?			

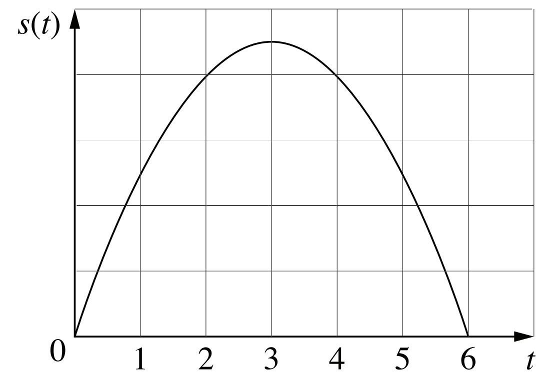
(1d) From the table, what could you observe about $\frac{s(t)-s(1)}{t-1}$? Give a physical interpretation for your observation.

(1e) The above observation, we say that instantaneous velocity $v(1)$ of the ball at $t = 1$ second is given by the

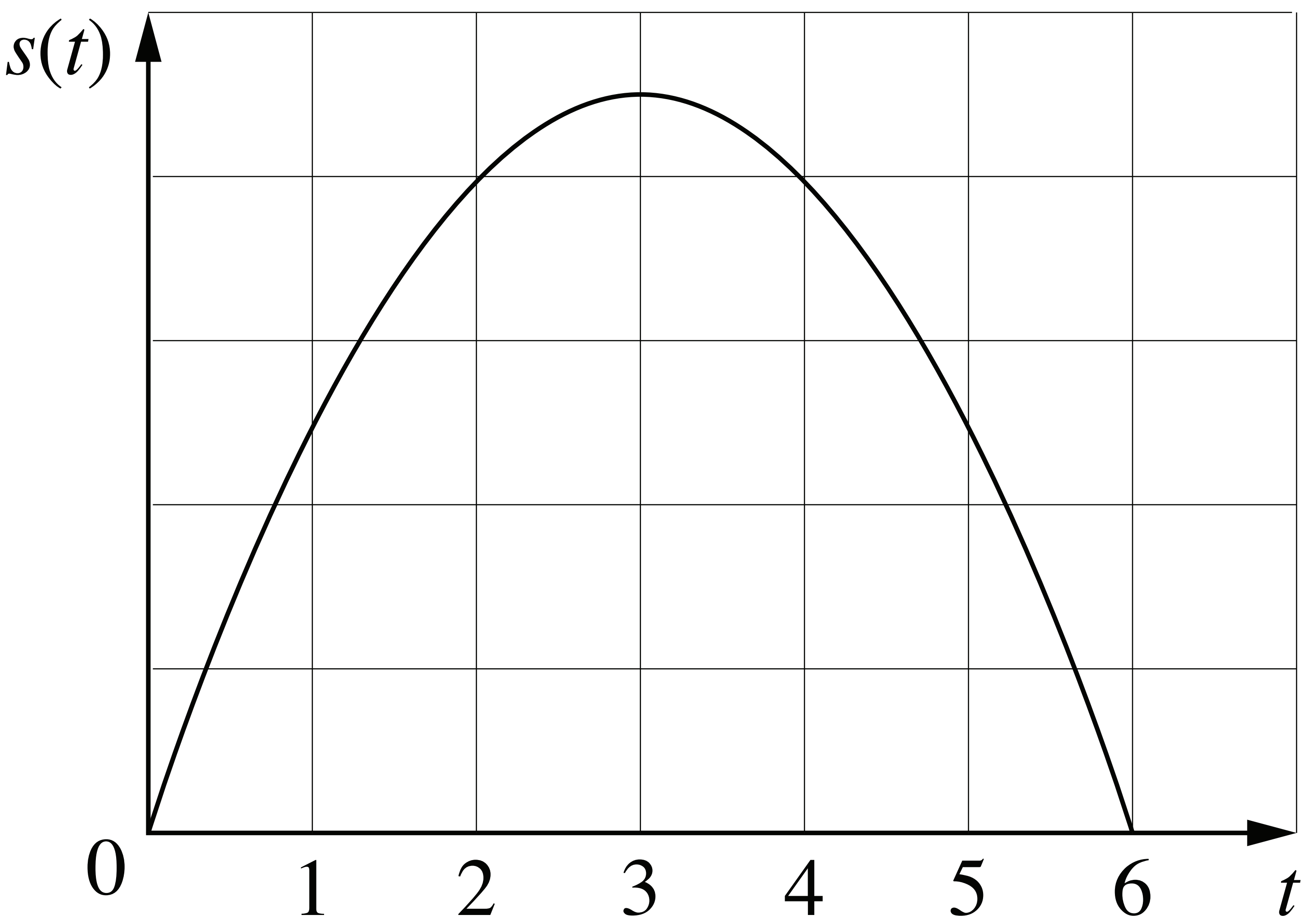
____________________ of the average velocity $\frac{s(t)-s(1)}{t-1}$ as time t __________________________.

This denoted by $v(1) = s'(1) =$ __________________________.

(1f) Give a graphical interpretation of the average velocity of the ball over the time interval $1 \leq t \leq 4$. Of course, we can also interpret the average velocity over the time interval between 1 and any $t(\neq 1)$.



(1g) Give a graphical interpretation of the instantaneous velocity $v(1)$ of the ball at $t = 1$ second.



(1h) Find the equation of the tangent line to the graph of $s(t) = 30 - 5t^2$ at $t = 1$.

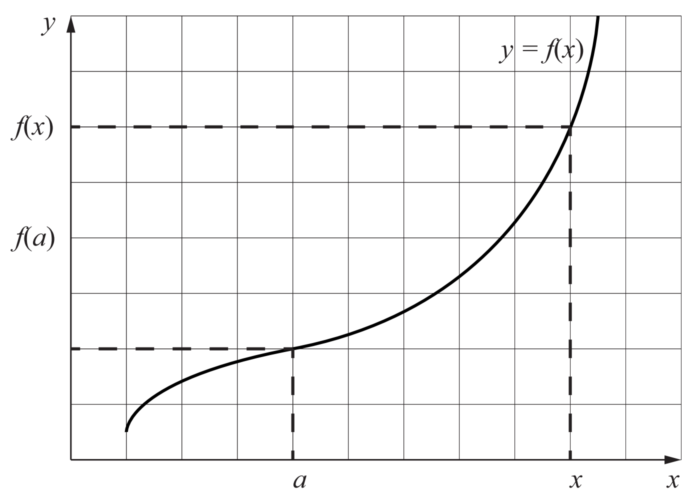
Summary. We have computed the instantaneous velocity at time $t = a$ of a particle moving on a straight line with position function $s(t)$. We online the key steps below.

Step 1: The average velocity of the particle over the time interval between t and a is ______________________________.

Step 2: The instantaneous velocity of the particle at $t = a$ is

$v(a) = s'(a) =$

The same limiting process above can be applied to many functions besides the position function of a particle. We can mimic the same limiting process to find the **Instantaneous Rate of Change of a function** $f(x)$ **at a given** $x = a$**.** Illustrate the process in the graph below.



Step 1: The average rate of change of $f(x)$ over the time interval between x and a is ______________________________.

Step 2: The instantaneous rate of change of $f(x)$ at $x = a$ is

$f'(a) =$

$f'(a)$ is also called the ____

and it gives the ______________________________________ to the graph of $f(x)$ at $x = a$.

2. Water is flowing into a tank at a rate such that the volume $V(t)$ (in cubic feet) of water in the tank at time $t \geq 0$ (in minutes) is given by $V(t) = \sqrt{t+4}$. Answer the following questions:

(a) Find the average rate of change of the volume of water over the time duration $[5, 12]$. What is the unit of your answer?

(b) Using limits, find the rate of change of the volume of water at the fifth minute. What is the unit of your answer?

Math 10350 – Example Set 03B
Sections 2.2, 2.3 & 2.4

Limit of a function. What happens to $f(x)$ as x gets as close to a fixed value c as we want? This question is answered with the concept of the limit of a function.

Explain what each of the following limits mean.

$L = \lim\limits_{x \to c^-} f(x)$ ____

We call this the ____________________ limit of $f(x)$ as x ____________________ c.

$L = \lim\limits_{x \to c^+} f(x)$ ____

We call this the ____________________ limit of $f(x)$ as x ____________________ c.

$L = \lim\limits_{x \to c} f(x)$ ____

We call this the (____________________) limit of $f(x)$ as x ____________________ c.

1. The graph of a function f is shown in Figure 1. By inspecting the graph, find each of the following values and limits if it exists. If the limit does not exist, explain why.

$\lim\limits_{x \to -1} f(x) \stackrel{?}{=}$ $\qquad f(-1) \stackrel{?}{=}$

$\lim\limits_{x \to -1^-} f(x) \stackrel{?}{=}$ $\qquad \lim\limits_{x \to -1^+} f(x) \stackrel{?}{=}$

$\lim\limits_{x \to 0} f(x) \stackrel{?}{=}$ $\qquad f(0) \stackrel{?}{=}$

$\lim\limits_{x \to 0^-} f(x) \stackrel{?}{=}$ $\qquad \lim\limits_{x \to 0^+} f(x) \stackrel{?}{=}$

$\lim\limits_{x \to 2} f(x) \stackrel{?}{=}$ $\qquad f(2) \stackrel{?}{=}$

$\lim\limits_{x \to 2^-} f(x) \stackrel{?}{=}$ $\qquad \lim\limits_{x \to 2^+} f(x) \stackrel{?}{=}$

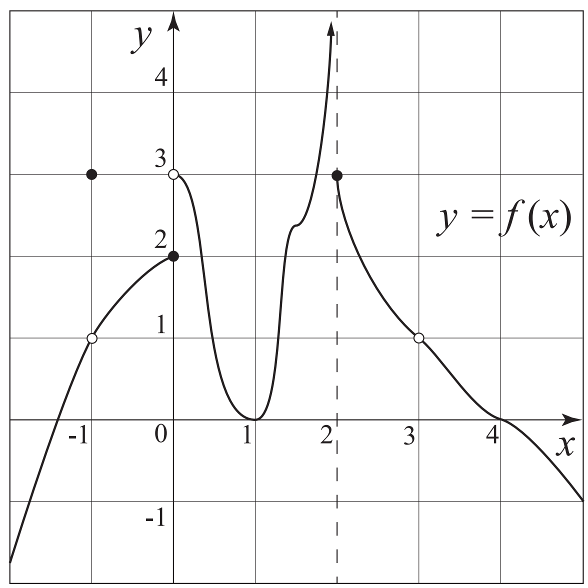


Figure 1

$\lim\limits_{x \to 3} f(x) \stackrel{?}{=}$ $\qquad f(3) \stackrel{?}{=}$

$\lim\limits_{x \to 3^-} f(x) \stackrel{?}{=}$ $\qquad \lim\limits_{x \to 3^+} f(x) \stackrel{?}{=}$

$\boxed{\textbf{Theorem 1}}$ (1) $\lim_{x \to c} f(x)$ exists $\iff$ ______________ and ______________ both <u>exist</u> and are <u>equal</u>.

Moreover, (2) $\lim_{x \to c} f(x) = L \iff$ ______________ $= L =$ ______________.

$\boxed{\textbf{Remark}}$ If any of the following are true:

$$\lim_{x \to c^-} f(x) = \infty; \qquad \lim_{x \to c^-} f(x) = -\infty; \qquad \lim_{x \to c^+} f(x) = \infty; \qquad \lim_{x \to c^+} f(x) = -\infty$$

then the graph of $f(x)$ has a ____________________________ at $x = c$.

$\boxed{\textbf{Definition}}$ (a) A function $f(x)$ is continuous at $x = c \iff f(c)$ is defined and __________________ $= f(c)$.

(b) A function $f(x)$ is **left** continuous at $x = c \iff f(c)$ is defined and __________________ $= f(c)$.

(c) A function $f(x)$ is **right** continuous at $x = c \iff f(c)$ is defined and __________________ $= f(c)$.

(d) A function $f(x)$ has a **jump** discontinuous at $x = c \iff$ __________________ $\neq$ __________________ .

(e) A function $f(x)$ has a **removable** discontinuous at $x = c \iff \lim_{x \to c} f(x)$ exist but __________________ $\neq f(c)$.

2. Comment on the continuity at $x = -1, 0, 1, 2$ for $f(x)$ in Figure 1. Are there any removable discontinuity?

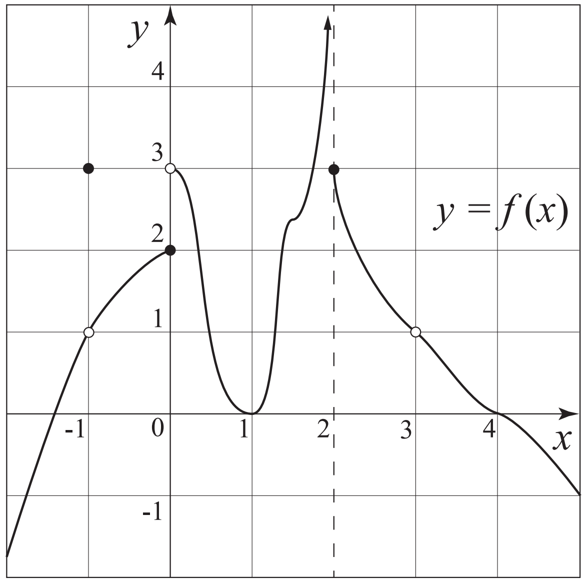


Figure 1

3. The graph of $f(x)$ is given in Figure 1 and $g(x) = 3x + 2$. By thinking about the values each function $f(x)$ and $g(x)$ approaches in the expressions below deduce the value of each limits:

(a) $\lim\limits_{x\to 3} [2f(x) + 3g(x)] \stackrel{?}{=}$

(d) $\lim\limits_{x\to 0} [f(x) - g(x)]^4 \stackrel{?}{=}$

(b) $\lim\limits_{x\to 2^+} [f(x) \cdot g(x)] \stackrel{?}{=}$

(e) $\lim\limits_{x\to 0} \sqrt{f(x)} \stackrel{?}{=}$

(c) $\lim\limits_{x\to 0^-} \dfrac{g(x)}{f(x) + 4} \stackrel{?}{=}$

Properties of Limits. Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then we have the following statements:

(1) $\lim_{x\to c} k \cdot f(x) =$

(2) $\lim_{x\to c} [f(x) + g(x)] =$

(3) $\lim_{x\to c} [f(x) - g(x)] =$

(4) $\lim_{x\to c} f(x) \cdot g(x) =$

(5) $\lim_{x\to c} \dfrac{f(x)}{g(x)} =$

provided $\lim_{x\to c} g(x) \neq 0$.

(6) $\lim_{x\to c} [f(x)]^n =$

(7) $\lim_{x\to c} \sqrt[n]{f(x)} =$

$\lim_{x\to c} f(x) \geq 0$ if n is even.

Math 10350 – Example Set 03C
Sections 2.2, 2.3 & 2.4
Computing Limits and Continuity

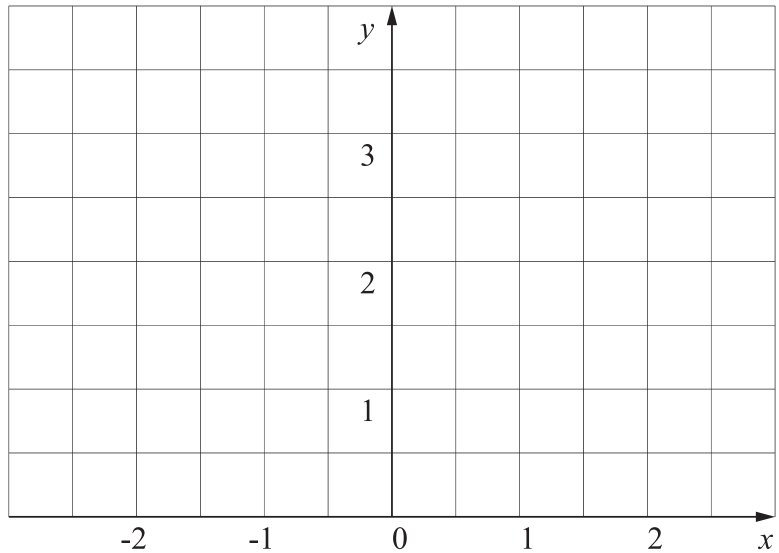
1. (Limit Concept) Consider the **piecewise defined** function $f(x) = \begin{cases} 2-x & -\infty < x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & 1 \le x < +\infty \end{cases}$

(a) Compute the following limits if they exist without sketching the graph.

i. $f(-1)$ **ii.** $f(3)$ **iii.** $\lim\limits_{x\to 1^-} [2f(x) - 5]$ **iv.** $\lim\limits_{x\to 0} (f(x) - 1)^2$ **v.** $\lim\limits_{x\to -5} (f(x))^2$

(b) Without sketching the graph of $f(x)$, determine the continuity of $f(x)$ at (i) $x = 0$ and (ii) $x = 1$.

(c) Sketch the graph of $f(x) = \begin{cases} 2 - x & -\infty < x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & 1 \leq x < +\infty \end{cases}$



2. Using limits find k such that the function $f(x)$ (a) is continuous at $x = 2$ and (b) is discontinuous at $x = 2$. Sketch a graph clearly depicting the nature of $f(x)$ at $x = 2$ in (a) and (b).

$$f(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2} & x \neq 2 \\ \\ k & x = 2 \end{cases}$$

3. Determine the constants a and b such that the following function is continuous on the entire real line

$$f(x) = \begin{cases} 2 & -\infty < x \leq -1 \\ ax^2 + b & -1 < x < 3 \\ -2 & 3 \leq x < +\infty \end{cases}$$