Math 10350 – Example Set 04A
Section 3.1

1. Consider the function

\[ f(x) = \begin{cases} 
5 - x & x < 2 \\
c & x = 2 \\
\frac{4}{x^2} & x > 2 
\end{cases} \]

Using limits describe the kind of discontinuity at \( x = 2 \). Without drawing the graph of \( f(x) \), find the value of \( c \) for which the function \( f(x) \) is (a) left continuous at \( x = 2 \), and (b) right continuous at \( x = 2 \).

**Definition 1** A function \( f(x) \) is said to be **differentiable** at \( x = c \) provided the following limit exist:

This means that the slope at \( x = c \) of the graph is a ________ number. We denote this number by \( f'(c) \).

Graphically, differentiable means that each small segment of the graph of \( f(x) \) is almost identical to a straight line. This is illustrated in Figure 1 through 3 below. As you zoom into the point \((c, f(c))\), the segment of the graph of \( f(x) \) near point \( c \) becomes more and more like its tangent line at \( x = c \).

**Remark:** We say that a function \( f(x) \) is differentiable on \((a, b)\) if \( f(x) \) is differentiable for all \( x = c \) in \((a, b)\).

**Theorem 1** If \( f(x) \) be differentiable at \( x = c \), then \( f(x) \) is ________ at \( x = c \).

2. Consider the function \( f(x) = \frac{1}{x} \).

a. Find the average rate of change of \( f(x) \) over the interval \( 2 \leq x \leq 5 \).

b. Find the average rate of change of \( f(x) \) over the interval \( 2 \leq x \leq (2 + h) \).

This is also called the (i) ___________________________ and (ii) ___________________________ at \( x = 2 \).

Simplify the expression as far as you can assuming that \( h \neq 0 \).

c. Using (b), find the derivative of \( f(x) \) at \( x = 2 \) using the limit definition.

d. What is the instantaneous rate of change of \( f(x) \) at \( x = 2 \)? ___________________________

e. What is the slope of the graph of \( f(x) \) at \( x = 2 \)? ___________________________
f. Find the equation of the tangent line to the graph of $f(x)$ at $x = 2$. Draw a graph that describe the limiting process in (c) and its connection to the tangent line.

g. Do the computation in Q2(b) and (c) replacing 2 by variable $x$ to obtain the derivative (slope function) of $f(x)$. Draw a picture to illustrate what the derivative represent.

**Derivative of a function.** The derivative of the function $f(x)$ is given by the following limit:

$$f'(x) = \frac{\Delta y}{\Delta x}$$

Setting $\Delta x = h$ and $\Delta y = f(x + h) - f(x)$ gives the notation:

$$f'(x) = \frac{f(x + h) - f(x)}{h}$$

**Notation:** If $y = f(x)$ is a differentiable function. Write down all standard notations of the derivative of $y = f(x)$.

**Some Common Derivatives.** For any numbers $k$ and $n$:

$$\frac{d}{dx}(k) = k$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(Power Rule)

**Basic Properties of Derivatives:**

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

$$[f(x) - g(x)]' = f'(x) - g'(x)$$

$$[c \cdot f(x)]' = c \cdot f'(x)$$

3. Find the derivative of each of the following functions:

a. $f(x) = \sqrt{x} + \frac{\pi}{\sqrt{x}}$

b. $y = \frac{x^3 + 5x + 6}{x}$

c. $h(t) = (2 + \sqrt{t}) t^2$
1. Recall the limit: \( \lim_{h \to 0} \frac{e^h - 1}{h} = \) \_. Use this to obtain formulas for \( \frac{d}{dx} (e^x) \) and \( \frac{d}{dx} (a^x) \).

2. The position (in feet) of a particle moving on a straight line is given by the function

\[
s(t) = \frac{5}{t} + t^e + 2e^t + 3t.
\]

Find an expression for the (instantaneous) velocity \( v(t) \). What is the velocity of the particle when \( t = \ln 2 \) seconds?

**Product and Quotient Rule.** Let \( f(x) \) and \( g(x) \) be differentiable functions. Derive formulas for the derivatives of \( p(x) = f(x) \cdot g(x) \) and \( q(x) = \frac{f(x)}{g(x)} \).

**Product Rule:**

\[
\frac{d}{dx} (f(x)g(x)) =
\]

**Quotient Rule:**

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) =
\]

3. Find the slope of the function \( f(x) = (x^2 + 2)e^x \) at \( x = 0 \). What is the equation of the tangent line there?

4. Water flows into a leaky container in such a way that the height of the water level from its base is given by \( h(t) = \frac{4e^t + 3}{e^t + 3} \) inches. (a) Find the time \( t \) such the height of the water level is 3 inches. (b) Find the instantaneous rate of change (ROC) of the height of the water when the water level reaches 3 inches.

5a. The stationary points in the domain of a function \( f(x) \) are the values of \( x \) such that \( f'(x) = 0 \). What can you say about the tangent line at stationary points?

5b. Find the stationary points of \( y = \frac{2x - 1}{x^2 + 1} \).
1. If \( f'(a) = \lim_{h \to 0} \frac{(3 + h)^{10} - 3^{10}}{h} \), what is a possible \( f(x) \) and the value of \( a \)?

\[ f(x) = \quad \text{and} \quad a = \]

2. The figure above describes the graph of \( y = f(x) \) and its tangent line at \( x = 3 \). Answer the problems below:

a. Estimate the average rate of change of \( f(x) \) over the interval \([0, 5]\).

b. \( f(3) = \) \quad \text{and} \quad f'(3) = \)

c. Find the equation of the tangent line at \( x = 3 \). Give your answer in slope-intercept form.

3. The slope of the curve \( y = ax^2 + bx \) at the point \((2, 4)\) is \(-8\). Calculate the values of \( a \) and \( b \).

b. Find \( \frac{dy}{dx} \bigg|_{x=-2} \) and use it to find the equation of the tangent line to the same curve at \( x = -2 \).

4. Find the values of \( x \) for which the graphs of the functions \( f(x) = x^3 - 3x^2 + 7x + 8 \) and \( g(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x - 3 \) have parallel tangent lines there. Pick one such location on the graph of \( f(x) \) and find the equation of the tangent line there.

5. Let \( p(x) = (x^3 - 5x + 1)g(x) \) and \( q(x) = \frac{f(x)}{g(x) + 1} \). Given that \( f(2) = 2 \), \( g(2) = 3 \), \( f'(2) = -1 \) and \( g'(2) = -4 \), find the following values:

a. The instantaneous rate of change of \( p(x) \) at \( x = 2 \).

b. The slope of the tangent line to the graph of \( y = q(x) \) when \( x = 2 \).
6. A military craft made with a new technology that could change its velocity on demand in a moment was test driven on a long straight road. The graph of its position $s(t)$ for eight seconds of travel is given below. Sketch in the given axes below the velocity function $v(t)$ indicating clearly places where velocity is undefined.