Math 10350 – Optimization on Closed and Bounded Interval

The extreme value theorem

If $f(x)$ is ________________________ on a closed and bounded interval $[a, b]$ then $f(x)$ takes on

a ________________________ and takes on a ________________________ on $[a, b]$.

On a closed and bound interval $[a, b]$, a continuous function $f(x)$ attains its absolute maximum and absolute minimum occur at the following possible locations

(1) ________________________ or

(2) ________________________ or

(3) ________________________.

Definition: Let $f(x)$ be defined at c. Then we say that c is a **critical point** of f if (A) __________________,

or (B) __________________.

Method for finding absolute maxima and minima of f on $[a, b]$

1. Find all critical points in (a, b).
2. Evaluate f at all critical points and at endpoints. Then compare the values of f:

   highest = absolute maximum and **lowest** = absolute minimum.

Section 4.2 Extreme Values

1. A piece of wire $10\,m$ long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is (a) maximum and (b) minimum.

Section 4.3 The Mean Value theorem

> **The Mean Value theorem**
> If $f(x)$ is continuous on $[a, b]$ and differentiable on the open interval (a, b) then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

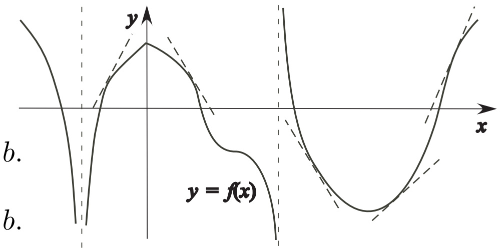
2. Verify that the function $f(x) = \dfrac{x}{x+2}$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$. Then find all numbers c that satisfies the conclusion of the Mean Value Theorem.

3. A police observed Kelly Brian's car passing his check point 8:00am at 62 mph. His patrolling buddy further down the 65mph-limit highway observed Kelly Brian's car 8:45am at 65 mph. Assuming that the highway is fairly straight and the two police are about 56 miles apart, can you conclude that Kelly Brian has been obeying the speed limit in the duration between 8am to 8:45am?

Math 10350 – Example Set 10B

Section 4.3: First Derivative and Monotonicity

Consider the graph of $f(x)$ below.



What does $f'(x)$ tell us about the graph of $f(x)$?

(1) If $f'(x) > 0$ for $a < x < b$, then $f(x)$ is _____________ for $a < x < b$.

(2) If $f'(x) < 0$ for $a < x < b$, then $f(x)$ is _____________ for $a < x < b$.

Remark: The only possible places (of x) where $f'(x)$ changes signs are at (i) ____________ or at (ii) where the graph has a ____________________ or undefined.

The First Derivative Test
Suppose $f(x)$ has a critical point at $x = c$. We classify the critical point as follows:
- if $f'(x)$ changes its sign from positive to negative at $x = c$, then there is a relative (local) ____________ at $x = c$.
- if $f'(x)$ changes its sign from negative to positive at $x = c$, then there is a relative (local) ____________ at $x = c$.
- if $f'(x)$ does not change its sign on both sides of $x = c$, then there is neither a relative (local) minimum nor a relative (local) maximum at $x = c$.

1. Find all values of x for which $f(x) = \dfrac{3}{4}x^{2/3} - x$ is increasing or decreasing with the steps outlined below. Classify all critical points using first derivative test.
Step 1: Find all **critical points** of f. (That is all points c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.)

Step 2: Find points where f have a **vertical asymptote** or undefined. Answer: _____________

Step 3: Draw a number line, mark all points found in Steps 1 and 2, and find the sign of $f'(x)$ in each intervals between marked points.

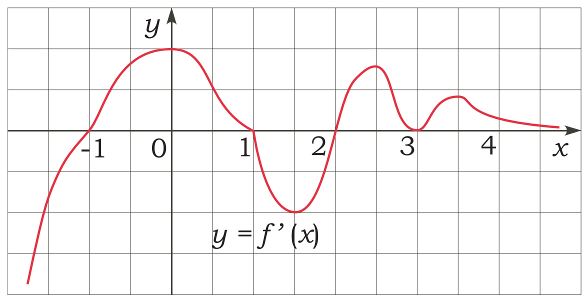
Step 4: Write down the values of x for which f is increasing ($f'(x) > 0$) and those for which f is decreasing ($f'(x) < 0$).

Step 5: Classify all critical points using first derivative test.

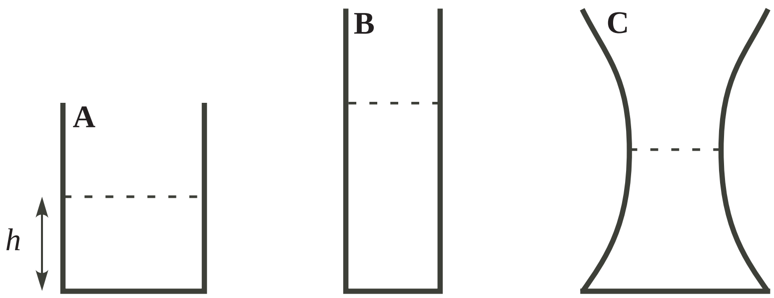
Math 10350 – Example Set 10C

Section 4.4: Second Derivative and Concavity

1. The graph of the **derivative** $f'(x)$ of $f(x)$ is given below. Find all critical points of $f(x)$ and use the derivative to determine where the function is increasing, where it is decreasing, and where it has a local maximum and minimum, if any.



2. Water is filling up each of the following vessels at a constant rate of 1 cm^3/sec.



Let h be the height of the water level in the vessel at time t.

a. Sketch the graphs of h verses t for Vessels A and B in the axes for Figure 1. Indicate which graph belong to A and which to B.


Figure 1

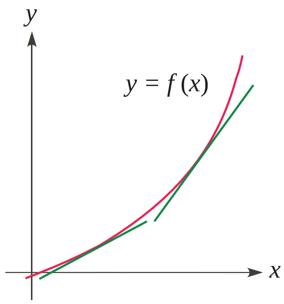

Figure 2

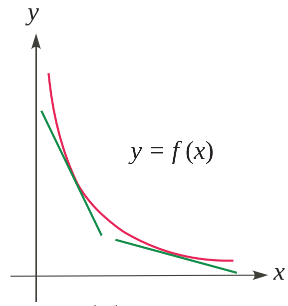
b. Sketch the graph of h verses time t for Vessel C in the axes for Figure 2.

c. Comment on how the "bending" (up or down) of the graph changes with $h'(t)$. Mark on the graph where the "bending" changes.

Characterization of Concavity

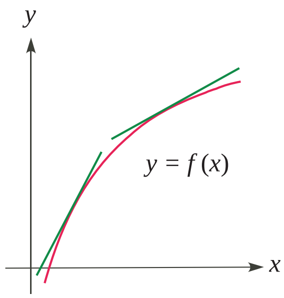
Case 1: For $a < x < b$, slope of the graph $f(x)$ is **increasing** as x increases i.e. $f'(x)$ is increasing. So $f''(x)$ is ____________ for $a < x < b$. (Portions of u-shape)

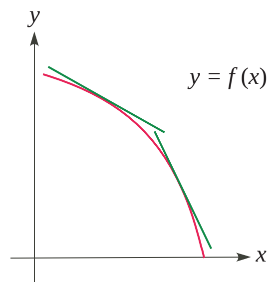




We say that the graph of $f(x)$ is ______________ for $a < x < b$.

Case 2: For $a < x < b$, slope of the graph $f(x)$ is **decreasing** as x increases i.e. $f'(x)$ is decreasing. So $f''(x)$ is ____________ for $a < x < b$. (Portions of n-shape)





We say that the graph of $f(x)$ is ______________ for $a < x < b$.

Definition (Inflection Points or Points of Inflection) We say that $(c, f(c))$ is a point of inflection of $f(x)$ if $f'(c)$ exist (so graph has tangent line at $x = c$), and the graph of $f(x)$ changes concavity at $x = c$.

Remark: Graph changes concavity at inflection point so to locate points of inflection we need to check the signum of the second derivative near these places in the domain:

(1) ____ , (2) ____

Note that the graph could also change its concavity at vertical asymptotes where neither $f(c)$, $f'(c)$ and $f''(c)$.

3. Find all points of inflection and concavity of $f(x) = \dfrac{9}{4}x^{4/3} - \dfrac{1}{6}x^3 + 3$.

Second Derivative Test
Let $f(x)$ be a function such that $f'(c) = 0$ and the function has a second derivative in an interval containing c.

- If $\boldsymbol{f''(c) > 0}$ then f has ____________________ at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) < 0}$ then f has ____________________ at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) = 0}$ then ____________________. Use first derivative test.

4. Find all relative extrema for the function $f(x) = \dfrac{9}{4}x^{4/3} - \dfrac{1}{6}x^3 + 3$. Use second derivative test to classify them whenever applicable. What would you do when the second derivative test does not apply at a critical point?