Some Common Derivatives. For any numbers k and n:

$\frac{d}{dx}(\sin(x)) \stackrel{?}{=}$

$\frac{d}{dx}(\cos(x)) \stackrel{?}{=}$

$\frac{d}{dx}(\tan(x)) \stackrel{?}{=}$

$\frac{d}{dx}(\cot(x)) \stackrel{?}{=}$

$\frac{d}{dx}(\sec(x)) \stackrel{?}{=}$

$\frac{d}{dx}(\csc(x)) \stackrel{?}{=}$

Solution for summary sheet

Implicit Differentiation. Can you find $\frac{dy}{dx}$ if $e^{xy} = x^3 - y^2 + 5$? Note $e^{xy(x)} = x^3 - (y(x))^2 + 5$

$$\frac{d}{dx}\left[e^{xy}\right] = \frac{d}{dx}\left[x^3 - (y(x))^2 + 5\right] \Rightarrow e^{xy}(xy' + 1 \cdot y) = 3x^2 - 2yy' + 0$$

$$\Rightarrow e^{xy} \cdot xy' + e^{xy} y = 3x^2 - 2yy' \Rightarrow xe^{xy} y' + 2yy' = 3x^2 - ye^{xy}$$

$$\Rightarrow y'(xe^{xy} + 2y) = 3x^2 - ye^{xy} \Rightarrow y' = \frac{3x^2 - ye^{xy}}{xe^{xy} + 2y}$$

Logarithmic Differentiation. For each of the derivatives below which one do you apply logarithmic differentiation? Find the derivative

$\frac{d}{dx}\left(e^{x^2+5}\right) \stackrel{?}{=} e^{x^2+5} \cdot (2x+0) = 2xe^{x^2+5}$

exponential type

$\frac{d}{dx}\left(x^{e^2+5}\right) \stackrel{?}{=} (e^2+5)x^{e^2+4}$

power function type

$\frac{d}{dx}\left(x^{x^2+5}\right) \stackrel{?}{=}$

Neither exponential function nor power function. Apply logarithmic function.

$$y = x^{x^2+5}$$

$$\ln y = \ln x^{x^2+5} \Rightarrow \ln y = (x^2+5) \cdot \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[(x^2+5)\ln x\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = (x^2+5) \cdot \frac{1}{x} + 2x \ln x$$

$$\frac{dy}{dx} = y\left[\frac{x^2+5}{x} + 2x \ln x\right] = x^{x^2+5} \cdot \left(\frac{x^2+5}{x} + 2x \ln x\right)$$

1. Find the equation of the tangent line to the graph of $y = \dfrac{2x+3}{4x+5}$ at $x = -1$.

One point: $x = -1$; $y = \dfrac{-2+3}{-4+5} = \dfrac{1}{1} = 1$

Slope at $x = -1$:

$$\frac{dy}{dx} = \frac{(4x+5)(2) - (2x+3)(4)}{(4x+5)^2} = \frac{8x+10-4x-12}{(4x+5)^2} = \frac{-2}{(4x+5)^2}$$

$$\text{Slope at } x=-1 = \left.\frac{dy}{dx}\right|_{x=-1} = \frac{-2}{(-4+5)^2} = -2$$

Equation of tangent at $x = -1$

$$y - 1 = -2(x-(-1))$$
$$= -2(x+1)$$

$$y = -2x - 1$$

2. Find the stationary points of the function $e^{\frac{x^3}{3}-\frac{x^2}{2}-6x+5}$. $\leftarrow f(x)$

$$f'(x) = \underbrace{e^{\frac{x^3}{3}-\frac{x^2}{2}-6x+5}}_{>0} \cdot (x^2-x-6) = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x+2)(x-3) = 0$$

$$\Rightarrow x = -2 \text{ or } 3$$

3. Find their derivatives of the following functions.

a. $y = (2x^2+5)^4$

$$\frac{dy}{dx} = 4(2x^2+5)^3(4x)$$

$$= 16x(2x^2+5)^3$$

b. $y = 4^{x^2+3}$

$$\frac{dy}{dx} = 4^{x^2+3} \cdot \ln 4 \cdot 2x$$

$$= 2x \cdot 4^{x^2+3} \ln 4.$$

c. $y = \ln\left(\frac{2e^{2x}+3}{3e^{2x}+4}\right) = \ln(2e^{2x}+3) - \ln(3e^{2x}+4)$

$$y' = \frac{1}{2e^{2x}+3} \cdot 4e^{2x} - \frac{1}{3e^{2x}+4} \cdot 6e^{2x}$$

$$= \frac{4e^{2x}}{2e^{2x}+3} - \frac{6e^{2x}}{3e^{2x}+4}.$$

d. $y = (\sin(x) + \cos^2(2x))^3$

$$y' = 3(\sin(x) + \cos^2(2x))^2 \cdot$$

$$\times \left(\cos x + 2\cos(2x) \cdot (-\sin(2x) \cdot 2)\right)$$

$$= 3(\sin x + \cos^2 2x)^2 \cdot$$

$$\times \left(\cos x - 4\cos 2x \sin 2x\right).$$

e. $y = \tan^3(e^{3x^2+4}) = \left(\tan\left(e^{3x^2+4}\right)\right)^3$

$$\frac{dy}{dx} =$$

$$3\left(\tan\left(e^{3x^2+4}\right)\right)^2 \sec^2\left(e^{3x^2+4}\right) e^{3x^2+4} \cdot 6x$$

$$= 18x e^{3x^2+4} \tan^2\left(e^{3x^2+4}\right) \sec^2\left(e^{3x^2+4}\right)$$

f. $y = \sqrt[3]{e^{\sec(x)} + \sec(e^x)} = \left(e^{\sec x} + \sec(e^x)\right)^{1/3}$

$$\frac{dy}{dx} =$$

$$\frac{1}{3}\left(e^{\sec x} + \sec(e^x)\right)^{-2/3} \cdot$$

$$\left(e^{\sec x} \cdot \sec x \tan x + \sec(e^x)\tan(e^x) \cdot e^x\right)$$

4. Let $f(x)$ be a differentiable function such that $f(2) = 1$ and $f'(2) = -1$.

4a. Find the slope of the graph of $y = f(x)e^{f(x)}$ at $x = 2$.

$$y'(x) = f(x) \cdot e^{f(x)} f'(x) + f'(x) e^{f(x)}$$

Slope of the graph of f(x) at $x = 2$

$$= f(2) e^{f(2)} f'(2) + f'(2) e^{f(2)}$$

$$= 1 \cdot e^{1} \cdot (-1) + (-1) e^{1} = -2e$$

4b. Find instantaneous rate of change of $y = \dfrac{f(x)}{f(x)+3}$ at $x = 2$.

$$\frac{dy}{dx} = \frac{(f(x)+3)f'(x) - f'(x) f(x)}{(f(x)+3)^2}$$

$$= \frac{f(x)f'(x) + 3f'(x) - f'(x)f(x)}{(f(x)+3)^2} = \frac{3f'(x)}{(f(x)+3)^2}$$

Instantaneous rate of change of y at $x = 2$

$$= \left.\frac{dy}{dx}\right|_{x=2} = \frac{3f'(2)}{(f(2)+3)^2} = \frac{3(-1)}{(1+2)^2}$$

$$= \frac{-3}{9} = -\frac{1}{3}$$

5. Find the equation of the tangent line to the curve $x\cos(1+2xy) = 2y^3 - 2$ at the point $(0,1)$.

Slope at $(0,1)$: $\frac{d}{dx}\left(x\cos(1+2xy)\right) = \frac{d}{dx}\left(2y^3-2\right)$

$$x\cdot\left(-\sin(1+2xy)\cdot\left(0+2x\frac{dy}{dx}+2y\right)\right)+1\cdot\cos(1+2xy) = 6y^2\frac{dy}{dx}-0$$

$$-x\sin(1+2xy)\left(2x\frac{dy}{dx}+2y\right)+\cos(1+2xy) = 6y^2\frac{dy}{dx}$$

$$-2x^2\sin(1+2xy)\frac{dy}{dx} - 2xy\sin(1+2xy)+\cos(1+2xy) = 6y^2\frac{dy}{dx}$$

$$-2xy\sin(1+2xy)+\cos(1+2xy) = \left(6y^2+2x^2\sin(1+2xy)\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2xy\sin(1+2xy)+\cos(1+2xy)}{6y^2+2x^2\sin(1+2xy)}$$

Slope at $(0,1)$ $= \left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} = \frac{0+\cos(1)}{6} = \frac{1}{6}\cos 1$

Equation of tangent: $y = \frac{\cos 1}{6}x + 1$.

6. Find the derivative of y with respect to x if $y = \frac{5-x}{(2x-3)^3}$. Completely simplify your answer giving it in the form $\frac{Ax+B}{(2x-3)^n}$

$$\frac{dy}{dx} = \frac{(2x-3)^3(-1) - 3(2x-3)^2(2)(5-x)}{(2x-3)^6}$$

$$= \frac{(2x-3)^2\left[(2x-3)(-1)-6(5-x)\right]}{(2x-3)^6}$$

$$= \frac{-2x+3-30+6x}{(2x-3)^4} = \frac{4x-27}{(2x-3)^4}$$