Math 10360 Review for Exam 3

1. Let $f(x,y) = x^3y + \ln(y-x)$. Find the following limits:

(a) $\displaystyle\lim_{h\to 0}\frac{f(1+h,2)-f(1,2)}{h}$ (b) $\displaystyle\lim_{h\to 0}\frac{f(1,2+h)-f(1,2)}{h}$ Ans. (a) 5; (b) 2.

2. Let $f(x,y) = \dfrac{x}{1+y} + e^{xy}$. Find all first and second partial derivatives of f.

$\frac{\partial f}{\partial x} = \frac{1}{y+1} + ye^{xy}$; $\frac{\partial f}{\partial y} = -x(1+y)^{-2} + xe^{xy}$; $\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}$; $\frac{\partial^2 f}{\partial y^2} = 2x(1+y)^{-3} + x^2e^{xy}$; $\frac{\partial^2 f}{\partial x\partial y} = -(1+y)^{-2} + e^{xy} + xye^{xy} = \frac{\partial^2 f}{\partial y\partial x}$

3. Let $z = f(x,y)$ be a function, with $f(1,2) = 5$, $\frac{\partial f}{\partial x}(1,2) = 2$, $\frac{\partial f}{\partial y}(1,2) = -3$. (a) Estimate the value of $f(0.8, 2.1)$, (b) Find the elasticity coefficient of z relative to x at $(1,2)$ and the elasticity coefficient of z relative to y at $(1,2)$.

4. Let $f(x,y) = ye^x + 2x - 2y + 5$. Find $f(0,0)$ and use linear approximation to estimate $f(-0.05, 0.01)$.

Ans. $f(-0.05, 0.01) \approx 4.89$.

5. The population $p(t)$ (in thousands) of cheetah is modeled by the differential equation $\dfrac{dp}{dt} = p\left(1 - \dfrac{p}{4}\right)$. If the initial population is 3 thousand, find $p(t)$. You are required to solve the differential equation.

(Ans: $p(t) = \frac{12e^t}{1+3e^t} = \frac{12}{e^{-t}+3}$)

6. A cup of warm coffee at 80^oC is cooling off outdoors according to the equation

$$\frac{dy}{dt} = k(y-20)$$

where $y(t)$ is the temperature (in ^{o}C) of the coffee at time t minutes. If the temperature of the coffee is 60^oC after 10 minutes, what is (i) the value of k, and (ii) the temperature when $t = 15$ minutes?

(Ans: (i) $k = \frac{1}{10}\ln\left(\frac{2}{3}\right)$, (ii) $y(15) = 60\left(\frac{2}{3}\right)^{3/2} + 20$)

7. Solve the following initial value problem: $x^2y' = 2xy - 3$; $y(1) = -1$. (Ans: $y(x) = x^{-1} - 2x^2$)

8. The temperature at a point (x,y) on a table is $T(x,y)$, measured in degree Celsius. A bug crawls so that its position on the table after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + (t/3)$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3) = 4$ and $T_y(2,3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds? (Ans: 2 ^{o}C/s)

9. The temperature at a point (x,y) on a flat metal plate is given by

$$T(x,y) = \frac{60}{1+x^2+y^2}$$

where T is measured in ^{0}C and x, y in meters. Determine which is bigger: the sensitivity of the temperature T in the x-direction or in the y-direction at the point $(2,1)$. Discuss the elasticity of T at $(2,1)$.

10. Find the rate of change of z with respect to y if $10\cos(x+y+z) = xz^2 + x^2y$

11. $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial s}$ where $u = xyz^2$; $x = \dfrac{t}{s}$, $y = t^2 + 2s$ and $z = e^{st}$. Find u_t when $t = 1$ and $s = -1$.

12. A tank contains 1000 L of brine with 15 kg of dissolved salt. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept throughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes? (c) What is the limiting concentration? How would your answers change if the brine is drain away at a rate of (i) 5 L/min, and (ii) 20 L/min?

13. The body weight of an animal which initially weighs 10kg is modeled by:

$$\frac{dw}{dt} = -\ln\left(\frac{w}{5}\right); \qquad w(0) = 10$$

Use Euler's method with $\Delta t = 0.1$ to estimate $w(0.2)$

14. Let $P(t)$ be the immigrant population (in thousands) of a small country, where t is time in months since the beginning of this year. A demographic study has shown that $P(t)$ is a solution to the following initial value problem:

$$\frac{dP}{dt} = 0.4t \cdot P, \quad P(0) = 2$$

Use Euler's Method with 3 steps to estimate $P(1.5)$.

15. (a) Find **all** possible estimates for the value of $\frac{\partial f}{\partial x}(2,1)$ and $\frac{\partial f}{\partial y}(2,1)$ if $f(x,y)$ is given by the table below. (b) How would you estimate the $f(2.1, 1.2)$ using the central difference estimate the partial derivatives at $(2,1)$?

y \ x	**1.5**	**2.0**	**2.5**
0.0	36	35	34
1.0	38	37	35
2.0	44	42	38

(Top-left cell of the table: **)

Math 10360 Review for Exam 3 Answers

12a. $\frac{dy}{dt} = 0.3 - \frac{y}{100}$; $y(0) = 15$ so $y(t) = 30 - 15e^{-t/100}$

12b. $y(20) = 30 - 15e^{-0.2}$

12c. Concentration at time t, $c(t) = \frac{y(t)}{1000} = 0.03 - 0.015e^{-t/100}$. Therefore $\lim_{t\to\infty} c(t) = 0.03$

12 (i) a. $\frac{dy}{dt} = 0.3 - \frac{5y}{1000+5t} = 0.3 - \frac{y}{200-t}$; $y(0) = 15$ so $y(t) = 0.15(200+t) - \frac{3000}{(200+t)}$

12 (i) b. $y(20) = 0.15(220) - \frac{3000}{220} = 33 - \frac{150}{11} = \frac{213}{11}$

12 (i) c. Concentration at time t, $c(t) = \frac{y(t)}{1000+5t} = 0.03 - \frac{600}{(200+t)^2}$. Therefore $\lim_{t\to\infty} c(t) = 0.03$

12 (ii) a. $\frac{dy}{dt} = 0.3 - \frac{20y}{1000-10t} = 0.3 - \frac{2y}{100-t}$; $y(0) = 15$ so $y(t) = 0.3(100-t) - \frac{3(100-t)^2}{2000}$

12 (ii) b. $y(20) = 0.3(80) - \frac{3(80)^2}{2000} = 24 - \frac{3(6400)}{2000} = 24 - \frac{48}{5} = 14\frac{2}{5}$

12 (ii) c. Concentration at time t, $c(t) = \frac{y(t)}{1000-10t} = 0.03 - \frac{3(100-t)}{20000}$. The tank dries out at $t = 100$ minutes. Just before it dries out $c = 0.03$ so saltier than at the start. In fact, $c'(t) > 0$ implies the concentration is increasing and reaches 0.03 just before it dries out.

13. $w(0) = 10$; $w(0.1) \approx 9.930685282$; $w(0.2) \approx 9.862066124$.

14. $P(0) = 2$; $P(0.5) = 2$; $P(1) = 2.2$; $P(1.5) = 2.64$.

Math 10360: Calculus B
Exam III
April 21, 2050

Name: ______________________________

Class Time: __________________________

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

Honor pledge. "As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.":

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Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	a	b	c	d	e
2.	a	b	c	d	e
3.	a	b	c	d	e
4.	a	b	c	d	e
5.	a	b	c	d	e
6.	a	b	c	d	e
7.	a	b	c	d	e
8.	a	b	c	d	e
9.	a	b	c	d	e
10.	a	b	c	d	e

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

Name: ____________________________

Class Time: ________________________

Multiple Choice

1.(5 pts.) Let $P(K, L)$ be the production function for a manufacturing company where K is the amount of capital and L is the amount of labor. Suppose

$$P(20, 30) = 10; \qquad \frac{\partial P}{\partial K}(20, 30) = 2; \qquad \frac{\partial P}{\partial L}(20, 30)(20, 30) = -3$$

Using the linear approximation of $P(K, L)$ at $(20, 30)$, estimate the amount of production when $K = 20.5$ and $L = 29.5$.

(a) 7.5

(b) 12.5

(c) 11.5

(d) 8.5

(e) 10.5

2.(5 pts.) Find the second partial derivative $\frac{\partial^2 f}{\partial y \partial x}$ if $f(x, y) = \ln(3x + 2y)$.

(a) $\frac{6}{(3x + 2y)^2}$

(b) $\frac{-4}{(3x + 2y)^2}$

(c) $\frac{4}{(3x + 2y)^2}$

(d) $\frac{-9}{(3x + 2y)^2}$

(e) $\frac{-6}{(3x + 2y)^2}$

Name: ____________________________

Class Time: ________________________

3.(5 pts.) A 500-gallon tank contains 200 gallons of brine with concentration 1/4 pounds of salt per gallon. Brine containing 2/5 pounds of salt per gallon is pumped into the tank at a rate of 5 gallons per minute. The mixture is pumped out of the tank at a rate of 4 gallons per minute. If $y(t)$ is the amount of salt in the tank at time t, find the differential equation modeling the amount of salt in the tank at time t.

(a) $\dfrac{dy}{dt} = 2 - \dfrac{y}{50}$

(b) $\dfrac{dy}{dt} = 2 - \dfrac{4y}{200 - 4t}$

(c) $\dfrac{dy}{dt} = \dfrac{8}{5} - \dfrac{5y}{200 + t}$

(d) $\dfrac{dy}{dt} = 2 - \dfrac{4y}{200 - t}$

(e) $\dfrac{dy}{dt} = 2 - \dfrac{4y}{200 + t}$

4.(5 pts.) For the same scenario above, how long it take before the brine overflow?

(a) 100 minutes

(b) 75 minutes

(c) 300 minutes

(d) 500 minutes

(e) 60 minutes

Name: ____________________________

Class Time: ________________________

5.(5 pts.) Consider the solution curve of the differential equation below that passing through the point $(-1, 1)$.

$$y' = x^2 + y^2$$

Find the equation of the tangent line to the solution curve at $(-1, 1)$.

(a) $y = 2x - 3$

(b) $y = -2x - 1$

(c) $y = 2x + 3$

(d) $y = 2x$

(e) $y = 1$

6.(5 pts.) Let $f(x, y) = e^{x^2 y}$. Find the value of the limit

$$\lim_{h \to 0} \frac{f(2, -1 + h) - f(2, -1)}{h} .$$

(a) $2e^{-2}$

(b) $-2e^{-2}$

(c) Does not exist.

(d) e^{-4}

(e) $4e^{-4}$

Name: ______________________________

Class Time: __________________________

7.(5 pts.) Consider the initial value problem:

$$y' = 3x + y, \qquad y(0) = 1.$$

Using Euler's method with **TWO** steps of equal size estimate $y(0.2)$

(a) 1.24

(b) 4.3

(c) 1.1

(d) 1.424

(e) 2

8.(5 pts.) Find y in terms of t if

$$\ln(1-y) - \ln(2-y) = t.$$

(a) $y = \dfrac{1+2e^t}{1+e^t}$.

(b) $y = \dfrac{1+e^t}{1+2e^t}$.

(c) $y = \dfrac{2-e^t}{1-e^t}$.

(d) $y = \dfrac{1-2e^t}{1-e^t}$.

(e) $y = \dfrac{1-e^t}{2-e^t}$.

Name: ______________________________

Class Time: __________________________

9.(5 pts.) Let $u = 2x^3 - 3y^2$. If $x = \ln(t^2)$ and $y = t^2 - e^2$ find the rate of change of u with respect to t when $t = e$.

(a) $\dfrac{24}{e} - 2e$

(b) $\dfrac{48}{e}$

(c) $\dfrac{48}{e} - 12e^2$

(d) $\dfrac{48}{e} - 2e$

(e) $\dfrac{24}{e}$

10.(5 pts.) Solve the initial value problem:

$$\frac{dy}{dx} = (2y - 1)x; \qquad y(0) = 0$$

(a) $y = \dfrac{1}{2}(1 - e^{x^2})$

(b) $y = \dfrac{1}{2}(e^{x^2/2} - 1)$

(c) $y = \dfrac{1}{2}(e^{x^2} - 1)$

(d) $y = e^{x^2} - 1$

(e) $y = \dfrac{1}{2}(1 - e^{x^2/2})$

Name: ______________________________

Class Time: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(12 pts.) Find the partial derivative of z with respect to y if

$$\frac{x}{(2y+z)^2} = e^{y^2}z - 3z^4$$

Name: ____________________________

Class Time: ________________________

12.(12 pts.) Consider the initial value problem:

$$y' = (y-t)^2; \qquad y(-1) = 1$$

Use Euler's method with two equal steps to estimate $y(0)$.

Name: ______________________________

Class Time: __________________________

13.(12 pts.) Solve the following initial value problem:

$$(x^2+1)\left(\frac{dy}{dx}-3\right)=2xy; \qquad y(1)=0$$

Name: ____________________________

Class Time: ________________________

14.(12 pts.) [**Part A.**] Let $u = \sin(x^2 + y^2)$. Find $\dfrac{\partial u}{\partial s}$ if $x = st$ and $y = \dfrac{1}{s + 2t}$. Give your answer in terms of s and t.

[**Part B. (Unrelated to Part A)**] Discuss the elasticity of $z = e^{x^2+xy^2}$ relative to x and y at $(x, y) = (1, 1)$

Math 10360: Calculus B
Exam III
April 21, 2050

Name: ______________________________

Class Time: ANSWERS

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5.	a	b	•	d	e
6.	a	b	c	d	•
7.	•	b	c	d	e
8.	a	b	c	•	e
9.	a	•	c	d	e
10.	•	b	c	d	e

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

Name: ______________________________

Class Time: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(12 pts.) Find the partial derivative of z with respect to y if

$$\frac{x}{(2y+z)^2} = e^{y^2} z - 3z^4$$

$$\frac{\partial}{\partial y}\left[x\,(2y+z(x,y))^{-2}\right] = \frac{\partial}{\partial y}\left[e^{y^2} z(x,y) - 3(z(x,y))^4\right]$$

$$-2x\cdot(2y+z(x,y))^{-3}\cdot\left(2+\frac{\partial z}{\partial y}\right) = e^{y^2}\cdot\frac{\partial z}{\partial y} + 2ye^{y^2} z(x,y) - 3\cdot 4(z(x,y))^3\cdot\frac{\partial z}{\partial y}$$

$$-4x(2y+z)^{-3} - 2x(2y+z)^{-3}\frac{\partial z}{\partial y} = e^{y^2}\frac{\partial z}{\partial y} + 2ye^{y^2}z - 12z^3\frac{\partial z}{\partial y}$$

$$-4x(2y+z)^{-3} - 2ye^{y^2}z = 2x(2y+z)^{-3}\frac{\partial z}{\partial y} + e^{y^2}\frac{\partial z}{\partial y} - 12z^3\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{-4x(2y+z)^{-3} - 2ye^{y^2}z}{2x(2y+z)^{-3} + e^{y^2} - 12z^3}$$

<u>OR</u> $$\frac{4x(2y+z)^{-3} + 2ye^{y^2}z}{12z^3 - 2x(2y+z)^{-3} - e^{y^2}}$$

<u>OR</u> $$\frac{4x + 2ye^{y^2}z(2y+z)^3}{(12z^3 - e^{y^2})(2y+z)^3 - 2x}$$

Name: ______________________________

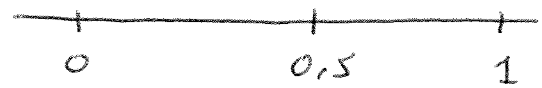
Class Time: __________________________

12.(12 pts.) Consider the initial value problem:

$$y' = (y+t)^2; \qquad y(0) = 1$$

Use Euler's method with two equal steps to estimate $y(1)$.

$\Delta t = \frac{1-0}{2} = 0.5$



$y(0) = 1$

$y(0.5) \simeq y(0) + y'(0) \cdot \Delta t$

$= 1 + 1(0.5) = 1.5$

$y(1) \simeq y(0.5) + y'(0.5)\,\Delta t$

$= 1.5 + (4)(0.5)$

$= 3.5$

$y'(t) = (y(t)+t)^2$

$y'(0) = (y(0)+0)^2$

$= 1$

$y'(0.5) = (y(0.5)+0.5)^2$

$= (1.5+0.5)^2 = 4$

Name: ______________________________

Class Time: __________________________

12.(12 pts.) Consider the initial value problem:

$$y' = (y-t)^2; \qquad y(-1) = 1$$

Use Euler's method with two equal steps to estimate $y(0)$.

Name: ____________________

Class Time: ____________________

13.(12 pts.) Solve the following initial value problem:

$$(x^2+1)\left(\frac{dy}{dx}-3\right)=2xy; \qquad y(1)=0$$

$$(x^2+1)y' - 3(x^2+1) = 2xy$$

$$(x^2+1)y' - 2xy = 3(x^2+1)$$

$$y' - \frac{2x}{x^2+1}\cdot y = 3$$

$$u = x^2+1$$
$$du = 2x\,dx$$

$$\text{Integrating factor} = e^{\int \frac{-2x}{x^2+1}dx}$$

$$= e^{\int -\frac{1}{u}du} = e^{-\ln u} = e^{\ln u^{-1}} = u^{-1} = (x^2+1)^{-1}$$

$$(x^2+1)^{-1}\cdot y' - 2x(x^2+1)^{-2}y = 3(x^2+1)^{-1}$$

$$\frac{d}{dx}\left[(x^2+1)^{-1}\cdot y\right] = \frac{3}{x^2+1}$$

$$\Rightarrow (x^2+1)^{-1}\cdot y = \int \frac{3}{x^2+1}dx = 3\arctan(x) + C$$

$$y(1)=0 \Rightarrow 2^{-1}(0) = 3\arctan(1) + C \Rightarrow C = -\frac{3\pi}{4}$$

$$(x^2+1)^{-1}\cdot y = 3\arctan(x) - \frac{3\pi}{4}$$

$$y = 3(x^2+1)\arctan(x) - \frac{3\pi}{4}(x^2+1).$$

Name: ______________________________

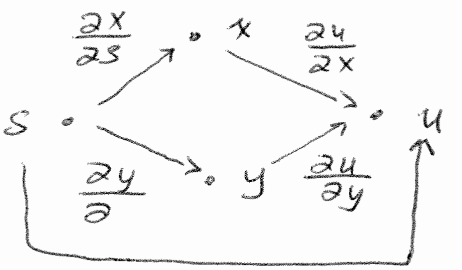
Class Time: ___________________________

14. (12 pts.) [**Part A.**] Let $u = \sin(x^2 + y^2)$. Find $\frac{\partial u}{\partial s}$ if $x = st$ and $y = \frac{1}{s+2t}$. Give your answer in terms of s and t.

$\leftarrow (s+2t)^{-1}$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x\cos(x^2+y^2) \cdot t + 2y\cos(x^2+y^2) \cdot \left[-(s+2t)^{-2} \cdot 1\right]$$



$$= 2st^2\cos\left[s^2t^2 + \frac{1}{(s+2t)^2}\right]$$

$$+ 2 \cdot \frac{1}{s+2t} \cos\left(s^2t^2 + \frac{1}{(s+2t)^2}\right) \cdot \frac{-s}{(s+2t)^2}$$

$$x^2+y^2 = s^2t^2 + \frac{1}{(s+2t)^2}$$

$$= \left(2st^2 - \frac{2s}{(s+2t)^3}\right)\cos\left(s^2t^2 + \frac{1}{(s+2t)^2}\right)$$

[**Part B. (Unrelated to Part A)**] Discuss the elasticity of $z = e^{x^2+xy^2}$ relative to x and y at $(x, y) = (1, 1)$

- Find the percentage change in z when x increases by 1% and y does not change.

$$\Delta x = \frac{1}{100} \times 1 = \frac{1}{100} \quad ; \quad \Delta y = 0 .$$

$$\frac{\Delta z}{z(1,1)} \times 100\% \leftarrow ? \quad ; \quad z(1,1) = e^2$$

$$\Delta z \simeq \frac{\partial z}{\partial x}(1,1) \cdot \Delta x + \underbrace{\frac{\partial z}{\partial y}(1,1) \cdot \Delta y}_{0}$$

$$\frac{\partial z}{\partial x}(1,1) = (2x+y^2)e^{x^2+xy^2}\Big|_{\substack{x=1\\y=1}} = 3e^2$$

$$\Delta z \simeq 3e^2 \cdot \frac{1}{100} = \frac{3e^2}{100}$$

$$\frac{\Delta z}{z(1,1)} \times 100\% = \frac{\frac{3e^2}{100}}{e^2} \times 100\% = \boxed{3\%}$$

A similar computation can be done for elasticity relative to y

$Z_y(x,y) = 2xye^{x^2+xy^2}$

$Z_y(1,1) = 2e^2$

Here $\Delta x = 0$ and $\Delta y = 0.01$

So ΔZ is approx $2e^2/100$

So elasticity relative to y is 2%