Polar Coordinates. For circular regions centered at the origin, it is often simpler to parametrize (label) the region in terms of polar coordinates (r distance from the origin, and θ angle measured from the positive half of the x-axis) instead of the cartesian (or rectangular) coordinates (x, y).

The relations between the polar coordinates (r, θ) of a given on the plane and its cartesian coordinates (x, y) are given by

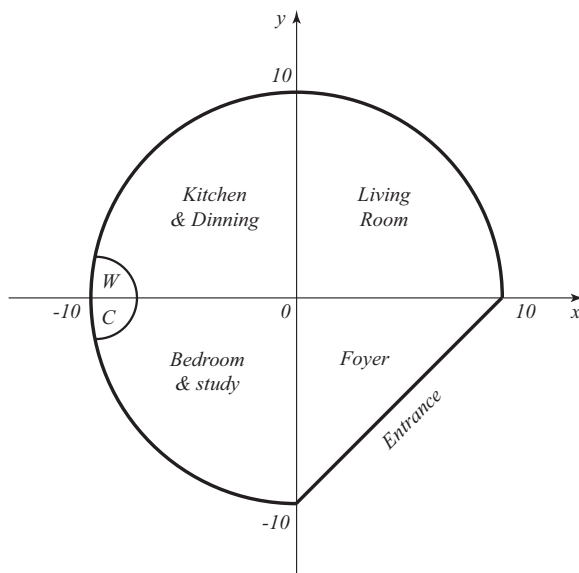$$x = r\cos(\theta); \qquad y = r\sin(\theta)$$

As a convention, we may restrict $r > 0$ and $0 \leq \theta < 2\pi$. Can you see the following the relations?

$$x^2 + y^2 = r^2; \qquad \tan\theta = \frac{y}{x}$$

1. Recall the hobbit house where the height of the house over each point on the floor (under the roof) is given by the function

$$f(x, y) = 40 - 2x + 2y \quad \text{meter}$$

Find the volume of the house enclosed by the roof over the kitchen, bedroom and living room area.

1. (15.4 Appln) The density of a quarter of a disc of radius 2m centered at the origin sitting in the first quadrant is given by the function

$$f(x, y) = 2x^2 + y^2 \text{ kg/m}^2.$$

Find the total mass of the quarter disc in kg.

2. (15.4 Appln) A variety of deep sea worm is distributed about a hydrothermal vent according to the population density

$$\rho(r, \theta) = \frac{8000 \sin^2(\theta/2)}{9 + r^2}$$

thousand per sq. miles where $1 \leq r \leq 3$ is the distance (in miles) from the vent. Find the total population of the sea worm.

Math 10360 – Example Set 08C
Separable Equations (9.1)

Separable Equations (9.1). Let $p(x)$ be a smooth function in x and $q(y)$ be a smooth function in y. Then a differential equation $\dfrac{dy}{dx} = F(x,y)$ is said to be separable if $F(x,y)$ can be written as a product of two smooth functions $p(x)$ and $q(y)$. That is $\dfrac{dy}{dx} = p(x) \cdot q(y)$.

Question: Is $\dfrac{dy}{dx} = \dfrac{p(x)}{q(y)}$ separable? What about $\dfrac{dy}{dx} = \dfrac{q(y)}{p(x)}$?

Separable Equations are solved by Method of Separation:

$$\frac{dy}{dx} = p(x) \cdot q(y) \iff y'(x) = p(x) \cdot q(y) \iff \frac{1}{q(y)} \cdot y'(x) = p(x).$$

Integrating both sides with respect to x gives: $\displaystyle \int \frac{1}{q(y)} \cdot y'(x)\, dx = \int p(x)\, dx.$

Note that $\dfrac{dy}{dx} = y'(x)$ ie $dy = y'(x)dx$. Thus we have: $\displaystyle \int \frac{1}{q(y)}\, dy = \int p(x)\, dx.$

Performing the integrals on the left and right hand sides give the an equation where we can usually find y in terms of x or, in general, an implicit relation between x and y.

Remark: One can skip some steps in the Method of Separation above by writing:

$$\frac{dy}{dx} = p(x) \cdot q(y) \iff \frac{1}{q(y)}\, dy = p(x)\, dx \iff \int \frac{1}{q(y)}\, dy = \int p(x)\, dx.$$

1. Solve for the general solution of $y'(x) = 3x^2 y$. Find the particular solution such that $y(0) = -2$.

Application to Autonomous Equations: $\dfrac{dy}{dt} = F(y)$

2. (9.1/9.2) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. A roast turkey is taken from an oven when its temperature is $165\ ^0F$ and is placed on a table in a room where the temperature is $70\ ^0F$. Temperature of the turkey falls to $150\ ^0F$ after half an hour. Apply Newton's Law of Cooling to find the temperature of the turkey after 45 minutes.