Math 10360 Example Set 15A
Sections 10.7 & 10.8 Taylor Polynomials & Taylor Series

A function $f(x)$ is said to be **analytic** if it has a (convergent) power series representation for each c i.e.

$$(1)\ f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots \qquad \text{for } -r < (x-c) < r.$$

where the coefficients a_i and radius of convergent r are to be determined. We can this series **The Taylor Series of the function** $f(x)$ centered at $x = c$

For the special case of $c = 0$, we get:

$$(2)\ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \qquad \text{for } -r < x < r.$$

We call (2) the **Maclaurin Series** for $f(x)$ or the Taylor Series for $f(x)$ centered at $x = 0$.

The geometric series summation formula give us an example of the Taylor series of $f(x) = \dfrac{1}{1-x}$ center at $x = 0$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1.$$

We will discuss in the next few lessons how to find the Taylor Series and its partial sums the Taylor Polynomials. The interval of convergence of Taylor Series are found using the Ratio Test.

1. (Formula for Taylor Series) Using repeated differentiation, show that the coefficients a_0, a_1, a_2, ... a_n, ... in the Taylor series for $f(x)$ centered at $x = c$:

$$a_0 = f(c), \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

This gives us the following formula for the Taylor series for $f(x)$ centered at $x = c$.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Remark: The Nth partial sum of the Taylor series is often used to estimate the value of $f(x)$. Specifically, we have:

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N$$

for x near to c.

The Nth partial sums of Taylor series for $f(x)$ centered at $x = c$ are called the **Nth Taylor Polynomials** of $f(x)$ centered at $x = c$. This polynomial is denoted $T_N(x)$. In the special case, when $c = 0$, we also call T_N the N-th **Maclaurin Polynomial** for $f(x)$.

(a) The 1st Taylor polynomial for $f(x)$ centered at c is $T_1(x) =$ ______________________.
(Linear Approximation of $f(x)$ at $x = c$)

(b) The 2nd Taylor polynomial for $f(x)$ centered at c is $T_2(x) =$ ______________________.

(c) The 3rd Taylor polynomial for $f(x)$ centered at c is $T_3(x) =$ ______________________.

We have the following theorem:

$\boxed{\textbf{Theorem.}}$ If $f(x)$ is analytic then there exists some r such that for some interval $c-r<x<c+r$ containing c, we have:

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

(a) For $c-r<x<c+r$ (especially for those x near c), we have the approximation

$$f(x)\approx T_n(x)=f(c)+\frac{f'(c)}{1!}(x-c)+\frac{f^{(2)}(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

(b) The accuracy of the approximation in (a) improves as n increases. More specifically,

$$f(x)=\lim_{n\to\infty}T_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

Remark: For the special case where $x=0$, the Taylor series for the function $f(x)$ centered at $x=0$ is also call the Maclaurin series for $f(x)$. This is simply the power series representation of $f(x)$ in x:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

1a. Find the Talyor Series centered at $x=0$ for $f(x)=e^x$. What is the interval of convergence for this power series?

1b. Using the Maclaurin polynomial $T_4(x)$ for e^x, estimate $e^{0.2}$.

1c. Write down the error of your estimate for $e^{0.2}$ in Q2(b) as a series. Explain your answer.

1d. Estimate the value of $\displaystyle\int_0^{0.2} e^{-x^2}\,dx$ using $T_4(x)$ for e^x.

2. Find the 3rd Taylor polynomial for the function $\ln(x+2)$ centered at -1, and estimate $\ln(0.8)$.

Math 10360 Example Set 15B

Application of Taylor polynomial and Taylor Series

1a. Find the 3rd-degree Taylor polynomial of $y(t)$ centered at zero, where $y(t)$ is the solution of the initial value problem

$$y' = y^2 + ty, \quad y(0) = -1$$

Use your result to estimate $y(0.3)$.

1b. Find the 3rd-degree Taylor polynomial of $y(t)$ centered at 1, where $y(t)$ is the solution of the initial value problem

$$y' = y^2 + ty, \quad y(1) = -1$$

Use your result to estimate $y(0.8)$.

2. Using the Taylor series for $\dfrac{1}{1+x^2}$ centered at $x = 0$ and differentiation, find the Maclaurin series for $\dfrac{2x}{(1+x^2)^2}$.

3a. Using the Taylor series for $\dfrac{1}{1+x^2}$ centered at $x = 0$ and integration, show that the Taylor series for $\arctan x$ at 0 is:

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad \text{for} \quad -1 < x < 1$$

3b. Write down the 7th Taylor series for $\arctan x$ at 0. Estimate the value of $\arctan(0.5)$. Write down the error for the estimate you found as an infinite series using summation notation.

3c. Write down the Taylor series for $f(x) = \arctan(1 - x)$ centered at 1, giving the values of x for which the series is convergent.

Math 10360 – Example Set 15C
Section 14.8

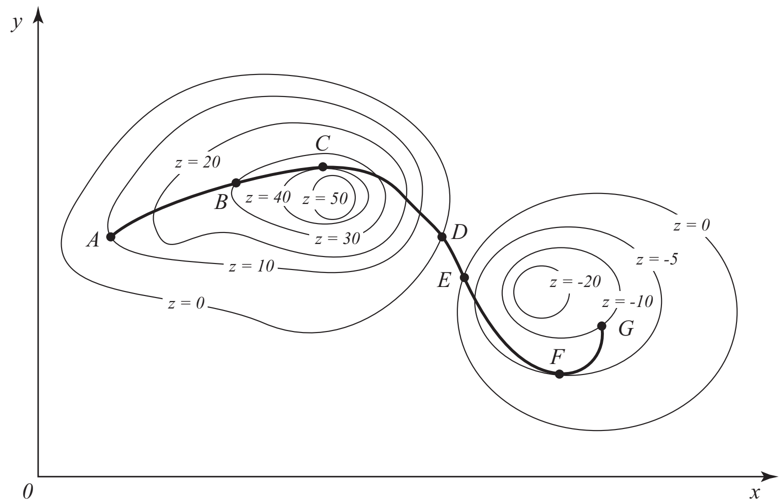
Topic: Optimization with a Constraint Using Lagrange Multipliers

Idea: Recall that for a continuous function $y = f(x)$ on a closed and bounded interval $a \le x \le b$, we optimize $f(x)$ with the following two facts:

(a) $f(x)$ must attain it minimum and maximum for some values of x in the interval $a \le x \le b$.

(b) The minimum and maximum of $f(x)$ occurs at the end points (i) $x = a, b$ or at (ii) critical points in $a < x < b$.

The range $a \le x \le b$ of values of x is the **constraint** on which $f(x)$ is optimized. However for multivariable functions the constraint may be some complicated relation satisfied by the indecent variables. For example, in the hiking exercise you did you are reading the highest point on your path above sea level and the lowest point on your path below sea level. In that context, the constraint is the hiking path on the $xy-$plane while the function you are optimizing is the height function.



Let the height function be given by $z = f(x, y)$ and the equation of the (projected) path be $g(x, y) = 0$.

From geometric considerations, we see that the constraint curve and the contour curve at a possible

min or max must share the same ____.

Therefore the critical points on the path are given by the equations:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \lambda \dfrac{\partial g}{\partial x}(x,y) & (1) \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = \lambda \dfrac{\partial g}{\partial y}(x,y) & (2) \\[2ex] g(x,y) = 0 & (3) \end{cases}$$

where x, y, and λ are to be determined. Here (x, y) are the critical points. Note that Equation (3) ensures the solution is on the constraint curve. The first two equations are called Lagrange Multipliers.

If the constraint path is closed and bounded (either a closed loop or curve including end points without self crossings) then the function $f(x, y)$ must attain minimum and maximum at some points on the path.

1a. The height of the slanted roof a house is given $h(x, y) = 2x + 4y + 20$. A spider on the roof is observed from the top view crawling on the closed path $x^2 + y^2 = 4$. What is the minimum and maximum height attained by the spider?

In this context, the function we need to optimize is the height $h(x, y) = 2x+4y+20$ with constraint $x^2+y^2-4 = 0$.

Can you roughly draw a picture to depict the path of the spider on the roof showing where the graph of $x^2 + y^2 = 4$ is in relation to the actual path of the spider? Use Lagrange multipliers to find the minimum and maximum heights of the spider.

1b. How would you change your answer if the spider only crawled on the path that tracks the upper semi-circular part of the curve $x^2 + y^2 = 4$?