Logarithmic Differentiation

Notes:

✓ Power function: x^n ← constant

← variable

$(x^n)' = nx^{n-1}$

← variable

✓ Exponential function: a^x

← constant

$(a^x)' = a^x \ln a$

← involves variable x

✓ $[f(x)]^{g(x)}$

← involves variable x

e.g. x^x ~ neither power function nor exponential fn.

→ Need Logarithmic differentiation to find its derivative.

Remark: We need to be able to identify the function type before we differentiate.

1. $y = \overset{\leftarrow \text{variable}}{x^{x^2}}$ ← neither power nor exponential.
↖ variable - apply log differentiation.

$\log a^x = x \log a$

$e^{\ln a} = a$

Wanted $\frac{dy}{dx}$.

Method 1:

$$\ln y = \ln(x^{x^2})$$

$$= x^2 \ln(x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x^2 \cdot \ln(x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x^2 \cdot \frac{1}{x} + 2x\ln(x)$$

$$= x + 2x\ln(x)$$

$$\frac{dy}{dx} = y(x + 2x\ln(x)) = x^{x^2}(x + 2x\ln(x)) \quad \#$$

Method 2:

$$e^{\ln a} = a$$

$$y = \underbrace{x^{x^2}}_{a} = \underbrace{e^{\ln x^{x^2}}}_{e^{\ln a}} = e^{x^2 \ln x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^2 \ln x}\right)$$ ← apply chain rule to exponential function

$$= \underbrace{e^{x^2 \ln x}} \cdot (x^2 \ln x)'$$

$$= x^{x^2} \cdot (x + 2x\ln x) \quad \#$$

variable

2. $y = (e+1)^{x} \longleftarrow$ exponential function.

constant

$$\frac{dy}{dx} = \underline{(e+1)^{x} \ln(e+1)}$$

constant

3. $y = x^{\pi^2} \longleftarrow$ power function.

variable

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\pi^2}\right)$$

$$= \underline{\pi^2 \cdot x^{\pi^2-1}}$$

4. $y = (3+4x)^{2x \leftarrow \text{variable}}$ ← neither power nor exponential function.

$\nwarrow$ variable

Method 2:

$$y = (3+4x)^{2x} = e^{\ln(3+4x)^{2x}}$$

$$= e^{2x\ln(3+4x)}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{2x\ln(3+4x)}\right)$$

$$= e^{2x\ln(3+4x)} \cdot \left(2x\ln(3+4x)\right)'$$

$$= \underbrace{e^{2x\ln(3+4x)}}_{} \cdot \left(2x \cdot \frac{4}{3+4x} + 2 \cdot \ln(3+4x)\right)$$

$$= (3+4x)^{2x}\left(\frac{8x}{3+4x} + 2\ln(3+4x)\right) \quad \swarrow \quad \checkmark$$

Method 1:

$$\ln y = \ln(3+4x)^{2x} = 2x \cdot \ln(3+4x)$$

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(2x \cdot \ln(3+4x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{8x}{3+4x} + 2\ln(3+4x) \quad \swarrow$$

$$\frac{dy}{dx} = \underline{\underline{y}}\left(\frac{8x}{3+4x} + 2\ln(3+4x)\right) = (3+4x)^{2x}\left(\frac{8x}{3+4x} + 2\ln(3+4x)\right)$$

5 $$y = \frac{(4+x)(2+3x)}{(1+2x)(3-2x)^3}$$ ← Rational function.

Quotient Rule is messy here.

$$\ln y = \ln\left(\frac{(4+x)(2+3x)}{(1+2x)(3-2x)^3}\right)$$

$\ln(ab) = \ln a + \ln b$

$$= \ln\left((4+x)(2+3x)\right) - \ln\left((1+2x)(3-2x)^3\right)$$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$$= \ln(4+x) + \ln(2+3x) - \ln(1+2x) - \ln(3-2x)^3$$

$$= \ln(4+x) + \ln(2+3x) - \ln(1+2x) - 3\ln(3-2x)$$

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\cdot\frac{dy}{dx}$$

$$= \frac{d}{dx}\left(\ln(\underline{4+x}) + \ln(\underline{2+3x}) - \ln(\underline{1+2x}) - 3\ln(\underline{3-2x})\right)$$

$$= \frac{1}{4+x} + \frac{3}{2+3x} - \frac{2}{1+2x} - 3\cdot\frac{-2}{3-2x}$$

$$= \frac{1}{4+x} + \frac{3}{2+3x} - \frac{2}{1+2x} + \frac{6}{3-2x}$$

$$\frac{dy}{dx} = y\left[\frac{1}{4+x} + \frac{3}{2+3x} - \frac{2}{1+2x} + \frac{6}{3-2x}\right]$$

$$= \frac{(4+x)(2+3x)}{(1+2x)(3-2x)^3}\left[\frac{1}{4+x} + \frac{3}{2+3x} - \frac{2}{1+2x} + \frac{6}{3-2x}\right]$$

$$= \frac{(2+3x)}{(1+2x)(3-2x)^3} + \frac{3(4+x)}{(1+2x)(3-2x)^3} - \frac{2(4+x)(2+3x)}{(1+2x)^2(3-2x)^3}$$

$$+ \frac{6(4+x)(2+3x)}{(1+2x)(3-2x)^4}.$$

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