Math 20480 Exam01 Review Name ______________________________

1. Perform the following operations involving matrices:

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} =$

(b) $2 \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^T =$

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^5 =$

2. It is known that the matrix $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ has the following eigenvalues and corresponding eigenvectors:

$$\lambda_1 = 2; \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3; \qquad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Find the general solution for the system of difference equations:

$$\begin{aligned} x(n) &= 4\,x(n-1) - 2\,y(n-1) \\ y(n) &= x(n-1) + y(n-1) \end{aligned}$$

3a. Does the matrix A has an inverse? If it does, find it.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ -2 & 3 & 1 \\ 4 & -2 & 7 \end{pmatrix}$$

3b. Solve the system of equations:

$$\begin{array}{rcrcrcl} x & + & y & + & 5z & = & 1 \\ -2x & + & 3y & + & z & = & 0 \\ 4x & - & 2y & + & 7z & = & 1 \end{array}$$

4. Find the values of k for which the following system of equations has a **unique** solution.

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & kz & = & 3 \\ x & + & ky & + & 3z & = & 2 \end{array}$$

5a. Find the eigenvalues of the following matrix:

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$

5b. For matrix A above, find a real matrix P which diagonalize it. Give also the associated diagonal matrix. Explain how you would find A^{10}. If $q(x) = x^5 - 4x^3 + 7$, evaluate $q(A)$.

6. Using row echelon reduction, solve the following system of equations:

$$\begin{array}{rcrcrcrcrcl} x & - & 3y & + & 2z & - & w & + & 2t & = & 2 \\ 3x & - & 9y & + & 7z & - & w & + & 3t & = & 7 \\ 2x & - & 6y & + & 7z & + & 4w & - & 5t & = & 7 \end{array}$$

7. (Aronld & Yokoyama) Suppose a particular species of salmon lives to four years of age. In addition, suppose that the survival rate of salmon in their first, second, and third years is 0.5%, 7%, and 15%, respectively. You also know that each female in the fourth age class produces 5,000 female offspring. The other age classes produce no offspring.

7a. Draw the lifecycle graph for the salmon weighting the directed edges with appropriate rates.

7b. If x_1, x_2, x_3, x_4 are functions representing the number of salmon at each age-stage, write down a system of difference equations that describe the population growth of the female salmon by age-stages.

7c. Write down the Leslie matrix for the salmon population.

8. Find the solution for the system of difference equations:

$$\begin{array}{rcl} x(n) = & 5\,x(n-1) + 6\,y(n-1); & x(0) = -1 \\ y(n) = & 3\,x(n-1) - 2\,y(n-1); & y(0) = 1 \end{array}$$

9. Let $A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix}$. Find A^{20} and B^{10}.

10. Does the following system of equations has a **unique** solution?

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & 3y & + & z & = & -1 \\ 2x & + & y & + & 3z & = & 0 \end{array}$$

11. Construct the Lefkovitch model for an animal with the given data below. You may assume that census is taken after eggs are laid, and that the fecundities are already adjusted to account for male and female ratio.

Stage	Description	Stage Duration (Years)	Annual Survivor Rates	Annual Fecundity
1	Eggs/Hatchlings	1	0.4	0
2	Juvenile - Type 01	2	0.6	0
3	Juvenile - Type 02	4	0.6	0
4	Sub-Adults	5	0.5	0
5	Adults - Type 01	15	0.4	150
6	Adults - Type 02	10	0.4	100
7	Seniors	> 40	0.7	0

It is estimated that the graduation rates between stages as follows:

Stage 02 to 03	Stage 03 to 04	Stage 04 to 05	Stage 05 to 06	Stage 06 to 07
20%	30%	10%	20%	20%

12. Using row reduction find the inverse of the following 3×3 matrix.

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

13. Find the eigenvalues and their corresponding eigenspaces for the matrices below. Also diagonalize each of the matrices.

(a) $M = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -5 & 5 & -2 \end{pmatrix}$ (b) $N = \begin{pmatrix} 1 & 3 & 3 \\ 0 & -2 & 0 \\ 3 & 3 & 1 \end{pmatrix}$

Answer: (a) $\lambda = -2,\ 1,\ 3$ (b) $\lambda = -2,\ -2,\ 4$. The eigenspaces answers are not unique. Below are possible answers:

(a) $E_{-2} = \left\{ \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} ; r \in \mathbb{R} \right\}$, $E_1 = \left\{ \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} ; s \in \mathbb{R} \right\}$, $E_3 = \left\{ \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} ; t \in \mathbb{R} \right\}$

(b) $E_{-2} = \left\{ \begin{pmatrix} -r-s \\ r \\ s \end{pmatrix} ; r, s \in \mathbb{R} \right\}$, $E_4 = \left\{ \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} ; s \in \mathbb{R} \right\}$

Answer for Q11.

System of difference equation for the Leftkovitch model:

$$\begin{aligned}
x_1(n) &= 7.5x_4(n-1) + 56x_5(n-1) + 32x_6(n-1) \\
x_2(n) &= 0.4x_1(n-1) + 0.48x_2(n-1) \\
x_3(n) &= 0.12x_2(n-1) + 0.42x_3(n-1) \\
x_4(n) &= 0.18x_3(n-1) + 0.45x_4(n-1) \\
x_5(n) &= 0.05x_4(n-1) + 0.32x_5(n-1) \\
x_6(n) &= 0.08x_5(n-1) + 0.32x_6(n-1) \\
x_7(n) &= 0.08x_6(n-1) + 0.7x_7(n-1)
\end{aligned}$$

Leftkovitch model in matrix form:

$$\begin{pmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ x_4(n) \\ x_5(n) \\ x_6(n) \\ x_7(n) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 7.5 & 56 & 32 & 0 \\ 0.4 & 0.48 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.12 & 0.42 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.18 & 0.45 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.08 & 0.32 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.08 & 0.7 \end{pmatrix} \begin{pmatrix} x_1(n-1) \\ x_2(n-1) \\ x_3(n-1) \\ x_4(n-1) \\ x_5(n-1) \\ x_6(n-1) \\ x_7(n-1) \end{pmatrix}$$

Leftkovitch Matrix:

$$L = \begin{pmatrix} 0 & 0 & 0 & 7.5 & 56 & 32 & 0 \\ 0.4 & 0.48 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.12 & 0.42 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.18 & 0.45 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.08 & 0.32 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.08 & 0.7 \end{pmatrix}$$

Math 20480 Exam 01 Review

1 (a) $\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{3\times 2} = \begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix}$

1 (b) $2\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^T$

$$= \begin{pmatrix} 2 & 0 & 2 \\ 0 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

1 (c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} 2^5 & 5\cdot 2^4 \\ 0 & 2^5 \end{pmatrix} = \begin{pmatrix} 32 & 80 \\ 0 & 32 \end{pmatrix}$

Q2. $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$$\underbrace{\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}}_{\vec{X}(n)} = \begin{bmatrix} 4x(n-1) - 2y(n-1) \\ x(n-1) + y(n-1) \end{bmatrix}$$

$$\vec{X}(n) = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

$$\vec{X}(n) = A\vec{X}(n-1) = \cdots = A^n \vec{X}(0)$$

$$A\underbrace{\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}}^{P} = \begin{bmatrix}2 & 6\\ 2 & 3\end{bmatrix} = \overbrace{\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}}^{P}\overbrace{\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}}^{D}$$

$$A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$$

$$\vec{X}(n) = PD^nP^{-1}\vec{X}(0)$$

$$= \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}2^n & 0\\ 0 & 3^n\end{bmatrix}\begin{bmatrix}s\\ t\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}s\cdot 2^n\\ t\cdot 3^n\end{bmatrix}$$

$$\begin{pmatrix}x(n)\\ y(n)\end{pmatrix} = s\cdot(2^n)\begin{pmatrix}1\\ 1\end{pmatrix} + t\,(3^n)\begin{pmatrix}2\\ 1\end{pmatrix}$$

$$x(n) = s\cdot 2^n + 2t\cdot 3^n$$

$$y(n) = s\cdot 2^n + t\cdot 3^n$$

3(a) $$\begin{vmatrix}1 & 1 & 5\\ -2 & 3 & 1\\ 4 & -2 & 7\end{vmatrix} = 1\begin{vmatrix}3 & 1\\ -2 & 7\end{vmatrix} - 1\begin{vmatrix}-2 & 1\\ 4 & 7\end{vmatrix} + 5\begin{vmatrix}-2 & 3\\ 4 & -2\end{vmatrix}$$

$$= (21+2) - (-14-4) + 5(4-12)$$

$$= 23 + 18 - 40 = 41 - 40 = 1 \neq 0.$$ 2

3(a)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 4 & -2 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2+2r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 5 & 11 & 2 & 1 & 0 \\ 0 & -6 & -13 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3+r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 5 & 11 & 2 & 1 & 0 \\ 0 & -1 & -2 & -2 & 1 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 1 \\ 0 & 5 & 11 & 2 & 1 & 0 \end{array}\right]$$

$$r_2/(-1) \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & -1 \\ 0 & 5 & 11 & 2 & 1 & 0 \end{array}\right] \xrightarrow{r_3-5r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & -1 \\ 0 & 0 & 1 & -8 & 6 & 5 \end{array}\right]$$

$$r_2-2r_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 18 & -13 & -11 \\ 0 & 0 & 1 & -8 & 6 & 5 \end{array}\right]$$

3

$$r_1-r_2-5r_3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 23 & -17 & -14 \\ 18 & -13 & -11 \\ -8 & 6 & 5 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

3(b)

$$\underbrace{\begin{bmatrix} 1 & 1 & 5 \\ -2 & 3 & 1 \\ 4 & -2 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 & -17 & -14 \\ 18 & -13 & -11 \\ -8 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = 23 - 14 = 9.$$

$$y = 18 - 11 = 7$$

$$z = -8 + 5 = -3$$

4. $$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

has unique solution if $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix} \neq 0.$

$$1\begin{vmatrix} 3 & k \\ k & 3 \end{vmatrix} - \begin{vmatrix} 2 & k \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 1 & k \end{vmatrix} \neq 0$$

$$-k^2+9-(6-k)-(2k-3) \neq 0$$

$$-k^2-k+6 \neq 0 \Rightarrow k^2+k-6 \neq 0$$

$$(k+3)(k-2) \neq 0 \Rightarrow k \neq 2 \text{ or } -3$$

So k can be any number except 2 or -3.

5

5(a) $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda)+18$$

$$= -20 - 5\lambda + 4\lambda + \lambda^2 + 18 = \lambda^2 - \lambda - 2 = 0$$

$$= (\lambda-2)(\lambda+1) = 0$$

Eigenvalues : $\lambda = -1, 2$

$B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$

$$|B - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda)+1$$

$$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \Rightarrow \lambda = -1, -1$$

Repeated eigenvalues.

$C = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

$$|C - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+2$$

$$= \lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = \frac{+4 \pm \sqrt{16-20}}{2} = 2 \pm i$$ (complex eigenvalues)

5(b) $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) + 18$

$= -20 - 5\lambda + 4\lambda + \lambda^2 + 18 = \lambda^2 - \lambda - 2 = 0$

$= (\lambda - 2)(\lambda + 1) = 0$

Eigenvalues : $\lambda = -1, 2$

Case 1 : $\lambda = -1$

$(A + I)\vec{u} = 0 \Rightarrow \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -3u_1 + 6u_2 = 0 \Rightarrow u_1 = 2u_2$

$\vec{u} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} = u_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; u_2 is any real number.

Case 2 : $\lambda = 2$

$(A - 2I)\vec{v} = 0 \Rightarrow \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-6v_1 + 6v_2 = 0 \Rightarrow v_2 = v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; v_1 is any real number

5(b) $A = P\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}}_{D}P^{-1}$

where $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$A^{10} = (PDP^{-1})^{10}$$

$$= PD^{10}P^{-1} \quad \text{since } PP^{-1} = I$$

$$= P\begin{pmatrix} (-1)^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}P^{-1}$$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

So $A^{10} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 2^{10} \\ 1 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2^{10} & -2+2^{11} \\ 1-2^{10} & -1+2^{11} \end{pmatrix}$$

Q5(b) Find $q(A)$ if $q(x) = x^5 - 4x^3 + 7$

$A = P\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}P^{-1}$; $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$; $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$q(A) = A^5 - 4A^3 + 7I_2$

$$A^n = P\begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-1)^n & 2^n \\ (-1)^n & 2^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-1)^n - 2^n & 2(-1)^{n+1} + 2^{n+1} \\ (-1)^n - 2^n & (-1)^{n+1} + 2^{n+1} \end{pmatrix}$$

$$A^5 = \begin{pmatrix} -2 - 2^5 & 2(1) + 2^6 \\ -1 - 2^5 & 1 + 2^6 \end{pmatrix} = \begin{pmatrix} -34 & 66 \\ -33 & 65 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -2 - 2^3 & 2(1) + 2^4 \\ -1 - 2^3 & 1 + 2^4 \end{pmatrix} = \begin{pmatrix} -10 & 18 \\ -9 & 17 \end{pmatrix}$$

$$q(A) = A^5 - 4A^3 + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -34 & 66 \\ -33 & 68 \end{pmatrix} - \begin{pmatrix} -40 & 72 \\ -36 & 68 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -34+40+7 & 66-72 \\ -33+36 & 65-68+7 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -6 \\ 3 & 4 \end{pmatrix}$$

(6) $\left[\begin{array}{ccccc|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{array}\right]$

$\begin{array}{l} r_2 - 3r_1 \\ r_3 - 2r_1 \end{array} \left[\begin{array}{ccccc|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 3 & 6 & -9 & 3 \end{array}\right]$

$r_3 - 3r_1 \left[\begin{array}{ccccc|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$z + 2w - 3t = 1 \Rightarrow z = -2w + 3t + 1.$$

$$x - 3y + 2z - w + 2t = 2$$

$$x = 3y - 2z + w - 2t + 2$$

$$= 3y - 2(-2w + 3t + 1) + w - 2t + 2$$

$$= 3y + 4w - 6t - \cancel{2} + w - 2t + \cancel{2}$$

$$= 3y + 5w - 8t$$

where y, w and t are free parameters

(6) $\begin{pmatrix} 3y + 5w - 8t \\ y \\ -2w + 3t + 1 \\ w \\ t \end{pmatrix}$

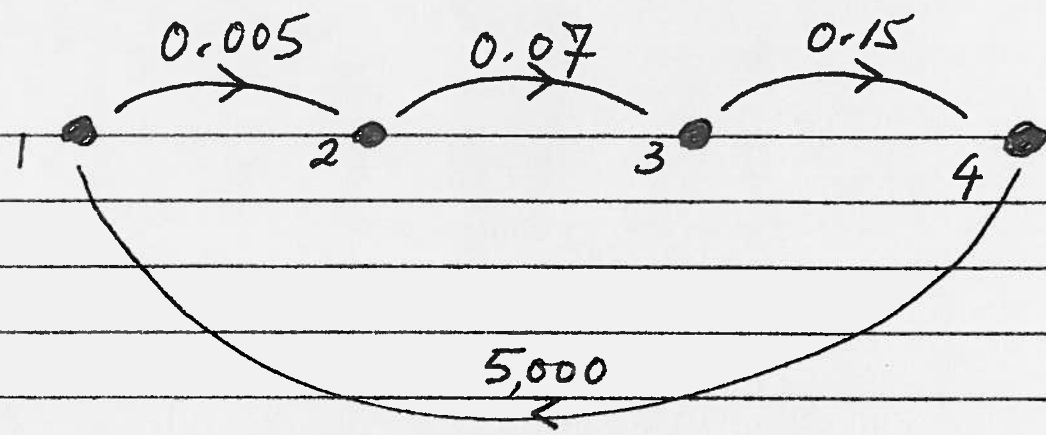
$$= \begin{pmatrix} 3y \\ y \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5w \\ 0 \\ -2w \\ w \\ 0 \end{pmatrix} + \begin{pmatrix} -8t \\ 0 \\ 3t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= y \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -8 \\ 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

y, w, t free parameters

(7)

(a)



(b)

$$x_1(n) = 5000\, x_4(n-1)$$

$$x_2(n) = 0.005\, x_1(n-1)$$

$$x_3(n) = 0.07\, x_2(n-1)$$

$$x_4(n) = 0.15\, x_3(n-1)$$

$$\begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ x_4(n) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5000 \\ 0.005 & 0 & 0 & 0 \\ 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \\ x_3(n-1) \\ x_4(n-1) \end{bmatrix}$$

(c) Leslie matrix $= \begin{bmatrix} 0 & 0 & 0 & 5000 \\ 0.005 & 0 & 0 & 0 \\ 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \end{bmatrix}$

Exam 01.
Math 20480 :

Q8

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} 5x(n-1) + 6y(n-1) \\ 3x(n-1) - 2y(n-1) \end{pmatrix}$$

$$\underbrace{}_{\vec{X}(n)} = \underbrace{\begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x(n-1) \\ y(n-1) \end{pmatrix}$$

So $\vec{X}(n) = A\vec{X}(n-1)$

Eigenvalues of A :

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 \\ 3 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda) - 18$$

$$= -10 + 2\lambda - 5\lambda + \lambda^2 - 18$$

$$= \lambda^2 - 3\lambda - 28 = (\lambda - 7)(\lambda + 4) = 0$$

$\Rightarrow \lambda = -4$ and 7.

Case 1: $\lambda = -4$

$$(A + 4I)\vec{u} = 0 \Rightarrow \begin{pmatrix} 9 & 6 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 9u_1 + 6u_2 = 0 \\ 3u_1 + 2u_2 = 0 \end{array}\right\} \Rightarrow u_2 = -\frac{3}{2}u_1$$

$$\vec{u} = \begin{pmatrix} u_1 \\ -3u_1/2 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$$

Eigenspace for $\lambda = -4$: $\left\{ s \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} ; -\infty < s < \infty \right\}$

Case 2: $\lambda = 7$

$$(A - 7I)\vec{v} = 0 \Rightarrow \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2v_1 + 6v_2 = 0 \\ 3v_1 - 9v_2 = 0 \end{cases} \Rightarrow v_1 = 3v_2$$

$$\vec{v} = \begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Eigenspace for $\lambda = 7$: $\left\{ t\begin{pmatrix} 3 \\ 1 \end{pmatrix} ; -\infty < t < \infty \right\}$

Pick $\vec{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ for $\lambda = -4$

$\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for $\lambda = 7$

$P = \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -4 & 0 \\ 0 & 7 \end{pmatrix}$.

So $A = PDP^{-1}$.

$$\vec{x}(n) = A\vec{x}(n-1) = \dots = A^n \vec{x}(0)$$

$$= PD^n P^{-1} x(0)$$

$$= \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} (-4)^n & 0 \\ 0 & 7^n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= a(-4)^n \begin{pmatrix} 2 \\ -3 \end{pmatrix} + b(7^n) \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$ $\leftarrow$ general solution

$$\vec{X}(0) = a\begin{pmatrix}2\\-3\end{pmatrix} + b\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\Rightarrow \quad 2a + 3b = -1 \sim (1)$$

$$-3a + b = 1 \sim (2)$$

$$(2) \times 3 : \quad -9a + 3b = 3 \sim (3)$$

$$(1) - (3) : \quad 11a = -4 \Rightarrow a = -4/11$$

$$\text{From } (2) : \quad \frac{12}{11} + b = 1 \Rightarrow b = -\frac{1}{11}.$$

$$\vec{X}(n) = -\frac{4}{11}(-4)^n\begin{pmatrix}2\\-3\end{pmatrix} - \frac{1}{11}(7^n)\begin{pmatrix}3\\1\end{pmatrix}$$

$$x(n) = \frac{2}{11}(-4)^{n+1} - \frac{3}{11}(7^n)$$

$$y(n) = -\frac{3}{11}(-4)^{n+1} - \frac{1}{11}(7^n).$$

from Q1.

Q.9 (a)

$$A^{20} = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}^{20} = (PDP^{-1})^{20}$$

$$= PD^{20}P^{-1} \qquad P = \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} (-4)^{20} & 0 \\ 0 & 7^{20} \end{bmatrix}\frac{1}{11}\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix} \qquad P^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 2(-4)^{20} & 3(7^{20}) \\ -3(-4)^{20} & (7^{20}) \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 2(-4)^{20} + 9(7^{20}) & -6(-4)^{20} + 6(7^{20}) \\ -3(-4)^{20} + 3(7^{20}) & 9(-4)^{20} + 2(7^{20}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{11}(-4)^{20} + \frac{9}{11}(7^{20}) & \frac{-6}{11}(-4)^{20} + \frac{6}{11}(7^{20}) \\ -\frac{3}{11}(-4)^{20} + \frac{3}{11}(7^{20}) & \frac{9}{11}(-4)^{20} + \frac{2}{11}(7^{20}) \end{bmatrix}$$

(b)

$$B^{10} = \begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix}^{10} = \left[-2\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\right]^{10}$$

$$= (-2)^{10}\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}^{10} = 2^{10}\begin{bmatrix} (-2)^{10} & 10(-2)^{9} \\ 0 & (-2)^{10} \end{bmatrix}$$

$$= 2^{10}\begin{bmatrix} 2^{10} & -10(2^{9}) \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 2^{20} & -10(2^{19}) \\ 0 & 2^{20} \end{bmatrix}$$

Write in matrix form:

Q10

$$\underbrace{\begin{pmatrix} 1 & 1 & -2 \\ 0^- & 3^+ & 1^- \\ 2 & 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\det(A) = 0 \cdot \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 0 + 3(3+4) - 1(1-2) = 21 + 1 \neq 0.$$

Yes, the given system of equation has a unique solution.

16

$[Q \mid I] \dashrightarrow [I \mid Q^{-1}]$

Q12

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Interchange rows to get the left matrix to have "pivots" at the best places and in this case to resemble I_3

Interchange rows $\longrightarrow$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$

$\xrightarrow{r_3 - r_1}$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$

$r_2 + r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$

Q^{-1}

Check:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$