Math 20480 Exam 02 Review Name ____________________

1. Suppose a 3×3 matrix A has eigenvalues 2, 3, and 3. Write down the canonical form associated to A in each of the following cases.

1a. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v} + n \cdot \vec{w}$ where k, m, and n are real constants.

1b. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v}$ where k, and m are real constants.

2. Suppose a 3×3 matrix A has eigenvalues -1, $3 \pm 4i$. Write down the canonical form associated to A. There are several possible answer depending on your construction, write down all of them.

3a. Find all the eigenvalues of $B = \begin{pmatrix} 2 & 0 & 0 \\ -5 & 0 & 5 \\ -2 & -1 & 4 \end{pmatrix}$ if one of them is 2.

3b. Still referring to B above. Write down the canonical form S associated to B and a matrix P such that $B = PSP^{-1}$.

4.

$$C = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

The matrix C above has eigenvalues -1, -1, and -1. We also know that the general form of the eigenvectors $\vec{u}$ of C and generalized eigenvectors $\vec{v}$ of C associated to $\vec{u}$ are given by

$$\vec{u} = \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} s \\ r \\ s \end{pmatrix}$$

where r and s are any real values.

4a. Find the formula for the generalized eigenvectors $\vec{w}$ associated to $\vec{v}$.

4b. Write down the Jordan form J associated to C and a matrix P such that $C = PJP^{-1}$.

5.

$$M = \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Matrix M has eigenvalues -1 and 2. Write down the canonical form associated to M.

6.

$$\Delta_A(\lambda) = \lambda^4 + 7\lambda^3 + 13\lambda^2 - 23\lambda - 78$$

The characteristic polynomial of matrix A is given above.

6a. What is the size of matrix A? Answer = ____________.

6b. If two roots of $\Delta_A(\lambda)$ are 2 and -3, find all the eigenvalues of A.

6c. Write down the canonical form associated to matrix A.

7. The eigenvalues of $B = \begin{pmatrix} 5 & 4 & -4 \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ is $\lambda =3, 3,$ and 3. Moreover the eigenvectors $\vec{u}$ and generalized eigenvector $\vec{w}$ of B are given below

$$\vec{u} = \begin{pmatrix} -2r+2s \\ r \\ s \end{pmatrix} \qquad\qquad \vec{w} = \begin{pmatrix} s+2t \\ 0 \\ t \end{pmatrix}$$

where r, s and t are any real number.

Solve the system of difference equations with given initial conditions

$$\begin{aligned} x(n) &= 5\,x(n-1) + 4\,y(n-1) - 4\,z(n-1) \qquad & x(0) &= 1 \\ y(n) &= 3\,y(n-1) \qquad & y(0) &= -1 \\ z(n) &= x(n-1) + 2\,y(n-1) + z(n-1) \qquad & z(0) &= -2 \end{aligned}$$

8.

$$C = \begin{pmatrix} 1 & -4 & 4 \\ 1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

The matrix C above has eigenvalues 3, 3, and 3. We also know that the eigenvectors are given by the formula

$$\vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}$$

where r is any real number.

8a. Write down the canonical form J associated to C.

8b. Find the generalized eigenvectors of matrix C.

8c. Write down the Jordan form J associated to C and a matrix P such that $C = PJP^{-1}$.

8d. Solve the system of difference equations with given initial conditions

$$\begin{aligned}
x(n) &= x(n-1) - 4\,y(n-1) + 4\,z(n-1) & x(0) &= 1 \\
y(n) &= x(n-1) + 4\,y(n-1) - z(n-1) & y(0) &= 2 \\
z(n) &= - y(n-1) + 4\,z(n-1) & z(0) &= 3
\end{aligned}$$

9.

$$M = \begin{pmatrix} -4 & -6 & 6 \\ -1 & 1 & 2 \\ -4 & -4 & 7 \end{pmatrix}$$

9a. The characteristic polynomial of M is $\lambda^3 - 4\lambda^2 + \lambda + 6$. Find all eigenvalues of M.

9b. Can M be diagonalized? If so write down the diagonal matrix J associated to M. If not, write down a more general canonical form J associated to M.

9c. For the canonical form J in 4(b), find the matrix P such that $M = PJP^{-1}$.

9d. Solve the system of difference equations:

$$\begin{aligned} x(n) &= -4\,x(n-1) - 6\,y(n-1) + 6\,z(n-1) \\ y(n) &= \,{-x(n-1)} + \,y(n-1) + 2\,z(n-1) \\ z(n) &= -4\,x(n-1) - 4\,y(n-1) + 7\,z(n-1) \end{aligned}$$

10.

$$Q = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -3 & 5 \\ -4 & -5 & 7 \end{pmatrix}$$

The eigenvalues and their associate eigenvectors of Q are given below

$$\lambda_1 = 2: \qquad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \qquad\qquad \lambda_1 = -1 + i: \qquad \vec{v} = \begin{pmatrix} 5s \\ (-1-2i)s \\ (2-i)s \end{pmatrix}$$

10a. Write down the remaining eigenvalue of Q and its associated eigenvectors.

10b. Write down the canonical form J associated to Q. Find also a matrix P such that $Q = PJP^{-1}$.

10c. Solve the system of difference equations:

$$\begin{aligned} x(n) &= \quad -4\,x(n-1) - 5\,y(n-1) + 5\,z(n-1) \\ y(n) &= \quad -2\,x(n-1) - 3\,y(n-1) + 5\,z(n-1) \\ z(n) &= \quad -4\,x(n-1) - 5\,y(n-1) + 7\,z(n-1) \end{aligned}$$

11a. Find the characteristic polynomial of matrix $T = \begin{pmatrix} 3 & -4 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 2 & 0 & 0 & -1 \end{pmatrix}$.

11b. If an eigenvalue of T is $1-2i$, find all the other eigenvalues. (Hint: Don't work too hard.)

12. Evaluate the given powers of the following matrices:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{pmatrix}^4 \stackrel{?}{=}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{pmatrix}^5 \stackrel{?}{=}$$

12 cont... Evaluate the given powers of the following matrices:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^4 \stackrel{?}{=}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^5 \stackrel{?}{=}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}^4 \stackrel{?}{=}$$

13. Consider the matrices: $Q = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $R = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $S = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$

Let $p(x) = 2 - 3x + x^5$. For any $n \times n$ matrix A, we define the matrix $p(A) = 2 \cdot I_n - 3A + A^5$. Find the following matrices:

13a. $p(Q)$

13b. $p(R)$

13c. $p(S)$

14. Evaluate the following complex numbers and express your answer in Euler's form. You should take the argument between 0 and 2π.

(a) $z_1 = \dfrac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$

(b) $z_2 = \left(\dfrac{1}{13} + \dfrac{5}{13}i\right) \cdot (3 + 2i)$

(c) $z_3 = (1 + i\sqrt{3})^{32}$

15. A study of a hermaphrodite organism concluded that there are four growth stages summarized in the table below

Stage	Description	Stage Duration (Years)	Annual Survivor Rates	Annual Fecundity
1	Eggs/Hatchlings	1	0.6	0
2	Juveniles	5	0.4	0
3	Young Adults	8	0.5	0
4	Adults	> 15	0.8	100

It is also known that **each year** 10% of those in Stage 2 will reach Stage 3, and 20% of those in Stage 3 will reach Stage 4. Answer the following questions assuming that census are taken right after nesting (eggs laying).

(a) Draw the lifecycle graph using the rates above. You should indicate the weights of the directed edges.

(b) If $x_1(n)$, $x_2(n)$, $x_3(n)$, and $x_4(n)$ are the number of the organism of stages 1, 2, 3, and 4 respectively in the nth year, write down the system of difference equations that describes the growth of the population of the organism according to its growth stages.

(c) Write down the Lefkovitch matrix L for the population model above.

16. The MatLab output for the Lefkovitch matrix M given in the attached Page 22 is for an animal having five growth stages: Eggs, Juveniles, Sub-Adult A, Sub-Adult B, and Adult.

(a) Estimate the continuous grow rate of the population after a long time. Is the population increasing or decreasing after a long time?

Continuous grow rate = ________________________

Circle One: Population Increases Population Decreases

(b) Find the stable population vector for the population. Your entries should be round to **four decimal places**.

(c) What is the population distribution in **percentages** after a long time? Give your answer round to **two decimal places**.

Eggs = ________________________

Juvenile stage = ________________________

Sub-adult A stage = ________________________

Sub-adult B stage = ________________________

Adult stage = ________________________

17. The MatLab information for a matrix A is given in the attached Page 11. Answer the following question about A

(a) Write down the canonical form C of A with all real entries.

(b) Find a real matrix P with all integer entries such that $A = PCP^{-1}$. Your entries for P should have **no unnecessary common factors.**

MatLab Output for Lefkovitch Matrix M in Question 16

M =

0	0	2.2500	39.2000	64.0000
0.4000	0.4500	0	0	0
0	0.0500	0.2550	0	0
0	0	0.0450	0.5600	0
0	0	0	0.1400	0.8000

```
>> [P, J] = jordan(M)
```

P =

4.0800 + 0.0000i	-1.0577 - 0.9203i	-1.0577 + 0.9203i	-0.0094 + 0.0000i	1.2714 + 0.0000i
-3.6267 + 0.0000i	1.7498 - 0.4864i	1.7498 + 0.4864i	-0.0306 + 0.0000i	1.0551 + 0.0000i
0.7111 + 0.0000i	0.1259 + 0.3274i	0.1259 - 0.3274i	-0.0048 + 0.0000i	0.0779 + 0.0000i
-0.0571 + 0.0000i	-0.0372 - 0.0184i	-0.0372 + 0.0184i	-0.0162 + 0.0000i	0.0094 + 0.0000i
0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i

J =

0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.2798 - 0.2577i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.2798 + 0.2577i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.5734 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.9320 + 0.0000i

MatLab Output for Matrix A in Question 17

A =

-1	-2	1	2
-3/8	-5/4	1/8	1/8
5/4	-5/2	9/4	5/4
4	-8	-2	2

```
>> [P , J ] = jordan(A)
```

P =

1/8 + 1/4i	1/8 - 1/4i	-3/4 + 0i	3/4 + 0i
0 + 0i	0 + 0i	-3/8 + 0i	0 + 0i
0 + 5/8i	0 - 5/8i	0 + 0i	0 + 0i
3/8 + 1/8i	3/8 - 1/8i	0 + 0i	-3/4 + 0i

J =

3 - 1i	0 + 0i	0 + 0i	0 + 0i
0 + 0i	3 + 1i	0 + 0i	0 + 0i
0 + 0i	0 + 0i	-2 + 0i	1 + 0i
0 + 0i	0 + 0i	0 + 0i	-2 + 0i

Math 20480 Exam 02 Review Name ______________________

1. Suppose a 3×3 matrix A has eigenvalues 2, 3, and 3. Write down the canonical form associated to A in each of the following cases.

1a. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v} + n \cdot \vec{w}$ where k, m, and n are real constants.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

There are three linearly independent eigenvectors. So enough to find matrix P to diagonalize A.

1b. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v}$ where k, and m are real constants.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

There only 2 linearly independent eigenvectors. Since ~~there~~ 3 ~~is~~ is a repeated, there must be a generalized eigenvector corresponding to $\lambda = 3$.

2. Suppose a 3×3 matrix A has eigenvalues -1, $3 \pm 4i$. Write down the canonical form associated to A. There are several possible answer depending on your construction, write down all of them.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}.$$

3a. Find all the eigenvalues of $B = \begin{pmatrix} 2 & 0 & 0 \\ -5 & 0 & 5 \\ -2 & -1 & 4 \end{pmatrix}$ if one of them is 2.

$$|B - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -5 & -\lambda & 5 \\ -2 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\left(-\lambda(4-\lambda) + 5\right)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 5) = (2-\lambda)(\lambda^2 - 4\lambda + 4 + 1)$$

$$= (2-\lambda)((\lambda-2)^2 + 1) = 0 \Rightarrow \lambda = 2, \quad (\lambda-2)^2 = -1$$

$$\lambda = 2 \pm i$$

Eigenvectors for $\lambda = 2$

$$(B - 2I)\vec{u} = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -5 & -2 & 5 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$-u_1 + u_3 = 0 \Rightarrow u_1 = u_3 \; ; \; u_2 = 0$$

$$\vec{u} = \begin{pmatrix} u_1 \\ 0 \\ u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

3b. Still referring to B above. Write down the canonical form S associated to B and a matrix P such that $B = PSP^{-1}$.

Eigenvector for $\lambda = 2 + i$: $(B - (2-i)I)\vec{v} = 0$

$$\left[\begin{array}{ccc|c} -i & 0 & 0 & 0 \\ -5 & -2-i & 5 & 0 \\ -2 & -1 & 2-i & 0 \end{array}\right] \xrightarrow{i \cdot R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -5 & -2-i & 5 & 0 \\ -2 & -1 & 2-i & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + 2R_1]{R_2 + 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2-i & 5 & 0 \\ 0 & -1 & 2-i & 0 \end{array}\right] \Rightarrow \begin{aligned} v_1 &= 0 \\ -v_2 + (2-i)v_3 &= 0 \\ v_2 &= (2-i)v_3 \end{aligned}$$

$$\vec{v} = \begin{bmatrix} 0 \\ (2-i)v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} v_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} i v_3$$

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \; ; \; \underline{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

4.

$$C = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

The matrix C above has eigenvalues -1, -1, and -1. We also know that the general form of the eigenvectors $\vec{u}$ of C and generalized eigenvectors $\vec{v}$ of C associated to $\vec{u}$ are given by

$$\vec{u} = \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} s \\ r \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + r\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where r and s are any real values.

4a. Find the formula for the generalized eigenvectors $\vec{w}$ associated to $\vec{v}$.

$$(C+I)\vec{w} = \vec{v} \Rightarrow \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} s \\ r \\ s \end{pmatrix}$$

$$-w_1 + w_3 = r \Rightarrow w_1 = w_3 - r$$

$$2w_1 + w_2 - 2w_3 = s \Rightarrow w_2 = s - 2w_1 + 2w_3 = s - 2(w_3 - r) + 2w_3$$
$$= s + 2r$$

$$\vec{w} = \begin{bmatrix} w_3 - r \\ s + 2r \\ w_3 \end{bmatrix} = w_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

4b. Write down the Jordan form J associated to C and a matrix P such that $C = PJP^{-1}$.

Pick $\vec{w} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ $(w_3 = 0 = s;\ r = 1)$

so $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}; \quad J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

5.

$$M = \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Matrix M has eigenvalues -1 and 2. Write down the canonical form associated to M.

$$|M - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -2 & 0 \\ 0 & -1-\lambda & 1 & 0 \\ 0 & -2 & 1-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & -1-\lambda & 1 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(2-\lambda)\left((-1-\lambda)(1-\lambda)+2\right)$$

$$= -(\lambda+1)(2-\lambda)(-1+\lambda^2+2)$$

$$= -(\lambda+1)(2-\lambda)(\lambda^2+1) = 0$$

$$\lambda = -1,\ 2,\ \pm i$$

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

6.

$$\Delta_A(\lambda) = \lambda^4 + 7\lambda^3 + 13\lambda^2 - 23\lambda - 78$$

The characteristic polynomial of matrix A is given above.

6a. What is the size of matrix A? Answer = 4×4.

6b. If two roots of $\Delta_A(\lambda)$ are 2 and -3, find all the eigenvalues of A.

$$\begin{array}{r|rrrrr}
2 & 1 & 7 & 13 & -23 & -78 \\
 & & 2 & 18 & 62 & 78 \\
\hline
-3 & 1 & 9 & 31 & 39 & \boxed{0} \leftarrow \text{check!} \\
 & & -3 & -18 & -39 & \\
\hline
 & 1 & 6 & +13 & \boxed{0} \leftarrow \text{check} &
\end{array}$$

$$\Delta_A(\lambda) = (\lambda-2)(\lambda+3)(\lambda^2+6\lambda+13) = 0$$

$$\lambda^2+6\lambda+13 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36-4(13)}}{2} \qquad \begin{array}{r} -52 \\ 36 \\ \hline -16 \end{array}$$

$$= -3 \pm 2i$$

6c. Write down the canonical form associated to matrix A.

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix} \text{ or } \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix}$$

There are few more possible answers, can you write ~~of~~ them down?

7. The eigenvalues of $B = \begin{pmatrix} 5 & 4 & -4 \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ is λ =3, 3, and 3. Moreover the eigenvectors $\vec{u}$ and generalized eigenvector $\vec{w}$ of B are given below

$$\vec{u} = \begin{pmatrix} -2r + 2s \\ r \\ s \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} s + 2t \\ 0 \\ t \end{pmatrix}$$

where r, s and t are any real number.

Solve the system of difference equations with given initial conditions

$$\begin{aligned} x(n) &= 5\,x(n-1) + 4\,y(n-1) - 4\,z(n-1) & x(0) &= 1 \\ y(n) &= 3\,y(n-1) & y(0) &= -1 \\ z(n) &= x(n-1) + 2\,y(n-1) + z(n-1) & z(0) &= -2 \end{aligned}$$

$$\vec{u} = r\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \; ; \quad \vec{w} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} s$$

$$\left.\begin{aligned} &\text{Pick } \vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (s = 1,\ t = 0) \\ &\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad (r = 0,\ s = 1) \end{aligned}\right\} \leftarrow \text{Jordan chain for } \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \vec{v}$$

$$\vec{u} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\underline{P} = \begin{pmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; ; \quad J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $B = PJP^{-1}$.

$$\vec{X}(n) = B\,\vec{X}(n-1) = B^n\,\vec{X}(0) = PJ^n \underbrace{P^{-1}\,\vec{X}(0)}_{\vec{c}}.$$

$$= P\begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & n3^{n-1} \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\vec{X}(n) = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \cdot 3^n \\ c_2 \cdot 3^n + c_3 \cdot n3^{n-1} \\ c_3 \cdot 3^n \end{bmatrix}$$

$$\vec{X}(n) = \begin{bmatrix} -2c_1 \cdot 3^n + 2c_2 \cdot 3^n + 2c_3 n 3^{n-1} + c_3 \cdot 3^n \\ c_1 \cdot 3^n \\ c_2 \cdot 3^n + c_3 \cdot n 3^{n-1} \end{bmatrix}$$

general solution.

$$\vec{X}(0) = \begin{pmatrix} -2c_1 + 2c_2 + c_3 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$c_1 = -1 \quad , \quad c_2 = -2$$

$$c_3 = 1 + 2c_1 - 2c_2 = 1 - 2 + 4 = 3$$

$$\Rightarrow x(n) = 2 \cdot 3^n - 4 \cdot 3^n + 6n 3^{n-1} + 3 \cdot 3^n$$

$$= 3^n + 6n 3^{n-1}$$

$$y(n) = -3^n$$

$$z(n) = -2 \cdot 3^n + 3n 3^{n-1}$$

8.

$$C = \begin{pmatrix} 1 & -4 & 4 \\ 1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

The matrix C above has eigenvalues 3, 3, and 3. We also know that the eigenvectors are given by the formula

$$\vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix} = r\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \leftarrow$$ only 1-dimensional so two generalized eigenvector needed

where r is any real number.

8a. Write down the canonical form J associated to C.

$$J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

8b. Find the generalized eigenvectors of matrix C.

$$(C - 3I)\vec{v} = \vec{u} \Rightarrow \left[\begin{array}{ccc|c} -2 & -4 & 4 & 0 \\ 1 & 1 & -1 & r \\ 0 & -1 & 1 & r \end{array}\right]$$

$$\xrightarrow[r_1/2]{r_2 + r_3} \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 1 & 0 & 0 & 2r \\ 0 & -1 & 1 & r \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{ccc|c} 0 & -2 & 2 & 2r \\ 1 & 0 & 0 & 2r \\ 0 & -1 & 1 & r \end{array}\right]$$

So $v_1 = 2r$, $-v_2 + v_3 = r \Rightarrow v = v - r.$

$$\vec{v} = \begin{bmatrix} 2r \\ v_2 \\ v_2 + r \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} r$$

$$(C - 3I)\vec{w} = \vec{v} \Rightarrow \left[\begin{array}{ccc|c} -2 & -4 & 4 & 2r \\ 1 & 1 & -1 & v_2 \\ 0 & -1 & 1 & v_2 + r \end{array}\right]$$

$$\xrightarrow[r_1/2]{r_2 + r_3} \left[\begin{array}{ccc|c} -1 & -2 & 2 & r \\ 1 & 0 & 0 & 2v_2 + r \\ 0 & -1 & 1 & v_2 + r \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{ccc|c} 0 & -2 & 2 & 2v_2 + 2r \\ 1 & 0 & 0 & 2v_2 + r \\ 0 & -1 & 1 & v_2 + r \end{array}\right]$$

$w_1 = 2v_2 + r$

$-w_2 + w_3 = v_2 + r \Rightarrow w_3 = v_2 + r + w_2$

$$\vec{w} = \begin{pmatrix} 2v_2 + r \\ w_2 \\ v_2 + r + w_2 \end{pmatrix} = v_2\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + w_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

8c. Write down the Jordan form J associated to C and a matrix P such that $C = PJP^{-1}$.

Pick $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad (v_2 = 0 = w_2 \; ; \; r = 1)$

so $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

8d. Solve the system of difference equations with given initial conditions

$$\begin{aligned} x(n) &= x(n-1) - 4y(n-1) + 4z(n-1) & x(0) &= 1 \\ y(n) &= x(n-1) + 4y(n-1) - z(n-1) & y(0) &= 2 \\ z(n) &= \qquad\qquad - y(n-1) + 4z(n-1) & z(0) &= 3 \end{aligned}$$

$$\vec{X}(n) = C\vec{X}(n-1) = C^n \vec{X}(0) = PJ^n \underbrace{P^{-1}\vec{X}(0)}_{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}^n \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & n3^{n-1} & \frac{1}{2}n(n-1)\cdot 3^{n-2} \\ 0 & 3^n & n3^{n-1} \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$n = 0$:
$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - r_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$c_1 = 2, \quad c_2 = 0$ and $c_3 + c_2 = 1 \Rightarrow c_3 = 1 - 0 = 1.$

Q8(d) Continue.....

$$\vec{X}(n) = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & n3^{n-1} & \frac{1}{2}n(n-1)3^{n-2} \\ 0 & 3^n & n3^{n-1} \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2\cdot 3^n + \frac{1}{2}n(n-1)3^{n-2} \\ n3^{n-1} \\ 3^n \end{pmatrix}$$

$$= \begin{bmatrix} 2n3^{n-1} + 3^n \\ 2\cdot 3^n + \frac{1}{2}n(n-1)3^{n-2} \\ 3\cdot 3^n + n3^{n-1} + \frac{1}{2}n(n-1)3^{n-2} \end{bmatrix}$$

9.

$$M = \begin{pmatrix} -4 & -6 & 6 \\ -1 & 1 & 2 \\ -4 & -4 & 7 \end{pmatrix}$$

9a. The characteristic polynomial of M is $\lambda^3 - 4\lambda^2 + \lambda + 6$. Find all eigenvalues of M.

$6: \ \pm 1, \pm 2, \pm 3, \pm 6$

$\underline{\lambda = -1}: \quad \Delta_M(-1) = -1 - 4 - 1 + 6 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & +5 & -6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\Delta_M(\lambda) = (\lambda+1)(\lambda^2 - 5\lambda + 6)$

$= (\lambda+1)(\lambda-2)(\lambda-3) = 0$

$\Rightarrow \lambda = -1, 2, 3.$

9b. Can M be diagonalized? If so write down the diagonal matrix J associated to M. If not, write down a more general canonical form J associated to M.

There are 3 distinct eigenvalues so there are 3 linearly independent eigenvectors.

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9c. For the canonical form J in 4(b), find the matrix P such that $M = PJP^{-1}$.

$\underline{\lambda = -1:}$ $(M + I)\vec{u} = 0 \Rightarrow \left[\begin{array}{ccc|c} -3 & -6 & 6 & 0 \\ -1 & 2 & 2 & 0 \\ -4 & -4 & 8 & 0 \end{array}\right]$

$$\xrightarrow[r_3(-4)]{r_1/(-3)} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ -1 & 2 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & & 0 \end{array}\right]$$

$4u_2 = 0 \Rightarrow u_2 = 0 \Rightarrow u_1 + 2u_2 - 2u_3 = 0 \Rightarrow u_1 = 2u_3$

$$\vec{u} = \begin{pmatrix} 2u_3 \\ 0 \\ u_3 \end{pmatrix} = u_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 2:}$ $(M - 2I)\vec{v} = 0 \Rightarrow \left[\begin{array}{ccc|c} -6 & -6 & 6 & 0 \\ -1 & -1 & 2 & 0 \\ -4 & -4 & 5 & 0 \end{array}\right]$

$$\xrightarrow{r_3/(-4)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 5/4 & 0 \end{array}\right] \xrightarrow[r_3 + r_1]{r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 0 \end{array}\right]$$

$v_3 = 0 \Rightarrow v_1 + v_2 - v_3 = 0 \Rightarrow v_2 = -v_1$

$$\vec{v} = \begin{pmatrix} v_1 \\ -v_1 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 3:}$ $(M - 3I)\vec{w} = 0 \Rightarrow \left[\begin{array}{ccc|c} -7 & -6 & 6 & 0 \\ -1 & -2 & 2 & 0 \\ -4 & -4 & 4 & 0 \end{array}\right]$

$$\xrightarrow{r_3/(-4)} \left[\begin{array}{ccc|c} -7 & -6 & 6 & 0 \\ -1 & -2 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[r_1 + 7r_3]{r_2 + r_3} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$-w_2 + w_3 = 0 \Rightarrow w_2 = w_3$

$w_1 + w_2 - w_3 = 0 \Rightarrow w_1 = 0$

$$\vec{w} = \begin{bmatrix} 0 \\ w_2 \\ w_2 \end{bmatrix} = w_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9d. Solve the system of difference equations:

$$\begin{aligned} x(n) &= -4\,x(n-1) - 6\,y(n-1) + 6\,z(n-1) \\ y(n) &= -x(n-1) + y(n-1) - 2\,z(n-1) \\ z(n) &= -4\,x(n-1) - 4\,y(n-1) + 7\,z(n-1) \end{aligned}$$

$$\vec{X}(n) = M\,\vec{X}(n-1) = M^n\,\vec{X}(0)$$

$$= P J^n P^{-1} \vec{X}(0)$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1(-1)^n \\ c_2(2^n) \\ c_3(3^n) \end{bmatrix}$$

$$x(n) = 2c_1(-1)^n + c_2 \cdot 2^n$$

$$y(n) = -c_2 \cdot 2^n + c_3 \cdot 3^n$$

$$z(n) = c_1(-1)^n + c_3 \cdot 3^n$$

10.

$$Q = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -3 & 5 \\ -4 & -5 & 7 \end{pmatrix}$$

The eigenvalues and their associate eigenvectors of Q are given below

$$\lambda_1 = 2: \qquad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \qquad\qquad \lambda_2 = -1+i: \qquad \vec{v} = \begin{pmatrix} 5s \\ (-1-2i)s \\ (2-i)s \end{pmatrix}$$

10a. Write down the remaining eigenvalue of Q and its associated eigenvectors.

$$\lambda_3 = \overline{\lambda_2} = -1-i \quad ; \quad \vec{w} = \overline{\vec{v}} = \begin{pmatrix} 5s \\ (-1+2i)s \\ (2+i)s \end{pmatrix}$$

10b. Write down the canonical form J associated to Q. Find also a matrix P such that $Q = PJP^{-1}$.

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$r = 1 : \vec{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} ; \quad s = 1 : \vec{v} = \begin{pmatrix} 5 \\ -1-2i \\ 2-i \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} i$$

$$P = \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix}$$

10c. Solve the system of difference equations:

$$\begin{aligned} x(n) &= -4\,x(n-1) - 5\,y(n-1) + 5\,z(n-1) \\ y(n) &= -2\,x(n-1) - 3\,y(n-1) + 5\,z(n-1) \\ z(n) &= -4\,x(n-1) - 5\,y(n-1) + 7\,z(n-1) \end{aligned}$$

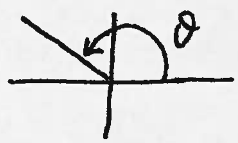
$$\vec{X}(n) = Q\,\vec{X}(n-1) = Q^n\,\vec{X}(0) = PJ^nP^{-1}\,\vec{X}(0)$$

Write $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ in the form $\begin{bmatrix} R\cos\theta & R\sin\theta \\ -R\sin\theta & R\cos\theta \end{bmatrix}$

$$\left.\begin{aligned} R\cos\theta &= -1 \rightsquigarrow (1) \\ R\sin\theta &= 1 \rightsquigarrow (2) \end{aligned}\right\} \Rightarrow \underline{(1)^2 + (2)^2}: \quad R^2\cos^2\theta + R^2\sin^2\theta = 1+1$$

$$R^2 = 2 \Rightarrow R = \sqrt{2}$$

$$\left.\begin{aligned} \sqrt{2}\cos\theta &= -1 \\ \sqrt{2}\sin\theta &= 1 \end{aligned}\right\} \Rightarrow$$



$$\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$$

$$\vec{X}(n) = \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2}\cos\left(\frac{3\pi}{4}\right) & \sqrt{2}\sin\left(\frac{3\pi}{4}\right) \\ 0 & -\sqrt{2}\sin\left(\frac{3\pi}{4}\right) & \sqrt{2}\cos\left(\frac{3\pi}{4}\right) \end{bmatrix}^n \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^{n/2}\cos\left(\frac{3\pi}{4}n\right) & 2^{n/2}\sin\left(\frac{3\pi}{4}n\right) \\ 0 & -2^{n/2}\sin\left(\frac{3\pi}{4}n\right) & 2^{n/2}\cos\left(\frac{3\pi}{4}n\right) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} C_1 2^n \\ C_2\cdot 2^{n/2}\cos(3\pi n/4) + C_3\cdot 2^{n/2}\sin(3\pi n/4) \\ -C_2\cdot 2^{n/2}\sin(3\pi n/4) + C_3\cdot 2^{n/2}\cos(3\pi n/4) \end{bmatrix}$$

$$x(n) = 5C_2\cdot 2^{n/2}\cos(3\pi n/4) + 5C_3\cdot 2^{n/2}\sin(3\pi n/4)$$

$$y(n) = C_1\cdot 2^n - (C_2 + 2C_3)\,2^{n/2}\cos(3\pi n/4) + (-C_3 + 2C_2)\cdot 2^{n/2}\sin(3\pi n/4)$$

$$z(n) = C_1\cdot 2^n + (2C_2 - C_3)\,2^{n/2}\cos(3\pi n/4) + (2C_3 + C_2)\cdot 2^{n/2}\sin(3\pi n/4)$$

11a. Find the characteristic polynomial of matrix $T = \begin{pmatrix} 3 & -4 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 2 & 0 & 0 & -1 \end{pmatrix}$.

$$\begin{array}{cc} & \begin{array}{cccc} + & - & & \end{array} \\ |T-\lambda I| = & \begin{vmatrix} 3-\lambda & -4 & 0 & 0 \\ 2 & -1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & -2 \\ 2 & 0 & 0 & -1-\lambda \end{vmatrix} \end{array}$$

$$= (3-\lambda)\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & 0 & -1-\lambda \end{vmatrix} + (4)\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 2 & 0 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda)(1-\lambda)(-1-\lambda) + 4(2)(1-\lambda)(-1-\lambda)$$

$$= (-1-\lambda)(1-\lambda)\left[(3-\lambda)(-1-\lambda) + 8\right] = (\lambda+1)(\lambda-1)\left(-3-2\lambda+\lambda^2+8\right)$$

$$= (\lambda+1)(\lambda-1)(\lambda^2-2\lambda+5)$$

11b. If an eigenvalue of T is $1-2i$, find all the other eigenvalues. (Hint: Don't work too hard.)

$$\lambda = 1,\ -1,\ 1-2i,\ 1+2i$$

12. Evaluate the given powers of the following matrices:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{pmatrix}^4 \stackrel{?}{=} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{8}\cos\pi/4 & \sqrt{8}\sin\pi/4 \\ 0 & 0 & -\sqrt{8}\sin\pi/4 & \sqrt{8}\cos\pi/4 \end{pmatrix}$$

Write $\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ as $\begin{pmatrix} R\cos\theta & R\sin\theta \\ -R\sin\theta & R\cos\theta \end{pmatrix} \Rightarrow$ $R\cos\theta = 2 \sim (1)$, $R\sin\theta = 2 \sim (2)$

$(1)^2 + (2)^2$: $R^2\cos^2\theta + R^2\sin^2\theta = 4+4 \Rightarrow R^2 = 8 \Rightarrow R = \sqrt{8}$

$(2)/(1)$: $\tan\theta = 1$ $\Rightarrow \theta = \pi/4$

$$\begin{pmatrix} 3^4 & 0 & 0 & 0 \\ 0 & 2^4 & 0 & 0 \\ 0 & 0 & 8^2\cos\pi & 8^2\sin\pi \\ 0 & 0 & -8^2\sin\pi & 8^2\cos\pi \end{pmatrix} = \begin{pmatrix} 81 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & -64 & 0 \\ 0 & 0 & 0 & -64 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{pmatrix}^5 \stackrel{?}{=} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2\cos\pi/6 & 2\sin\pi/6 \\ 0 & 0 & -2\sin\pi/6 & 2\cos\pi/6 \end{pmatrix}^5$$

Write $\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$ as $\begin{pmatrix} R\cos\theta & R\sin\theta \\ -R\sin\theta & R\cos\theta \end{pmatrix} \Rightarrow$ $R\cos\theta = \sqrt{3} \sim (1)$, $R\sin\theta = 1 \sim (2)$

$(1)^2 + (2)^2$: $R^2\cos^2\theta + R^2\sin^2\theta = 3+1 \Rightarrow R^2 = 4 \Rightarrow R = 2$

$(2)/(1)$: $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$$\begin{bmatrix} (-1)^5 & 5(-1)^4 & 0 & 0 \\ 0 & (-1)^5 & 0 & 0 \\ 0 & 0 & 2^5\cos 5\pi/6 & 2^5\sin 5\pi/6 \\ 0 & 0 & -2^5\sin 5\pi/6 & 2^5\cos 5\pi/6 \end{bmatrix}$$

$\cos 5\pi/6 = \frac{\sqrt{3}}{2}$

$\sin 5\pi/6 = \frac{1}{2}$

$$\begin{bmatrix} -1 & 5 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -16\sqrt{3} & 16 \\ 0 & 0 & -16 & -16\sqrt{3} \end{bmatrix}$$

12 cont... Evaluate the given powers of the following matrices:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^4 \stackrel{?}{=} \begin{bmatrix} 3^4 & 0 & 0 & 0 \\ 0 & 2^4 & 4\cdot 2^3 & 0 \\ 0 & 0 & 2^4 & 0 \\ 0 & 0 & 0 & 2^4 \end{bmatrix}$$

$$= \begin{bmatrix} 81 & 0 & 0 & 0 \\ 0 & 16 & 32 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^5 \stackrel{?}{=} \begin{bmatrix} (-1)^5 & 5(-1)^4 & \frac{1}{2}(5)(4)(-1)^3 \\ 0 & (-1)^5 & 5(-1)^4 \\ 0 & 0 & (-1)^5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & -10 \\ 0 & -1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}^4 \stackrel{?}{=} \begin{bmatrix} (-1)^4 & 4(-1)^3 & \frac{1}{\cancel{2}}(\overset{2}{\cancel{4}})(3)(-1)^2 & \frac{4!}{3!\,1!}(-1)^1 \\ 0 & (-1)^4 & 4(-1)^3 & \frac{1}{2}(4)(3)(-1)^2 \\ 0 & 0 & (-1)^4 & 4(-1)^3 \\ 0 & 0 & 0 & (-1)^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13 Consider the matrices: $Q = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $R = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $S = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$

Let $p(x) = 2 - 3x + x^5$. Find the following matrices:

13a. $p(Q) = 2I - 3Q + Q^5$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -9 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3^5 & 0 \\ 0 & (-1)^5 \end{pmatrix}$$

$$= \begin{pmatrix} 2-9+243 & 0 \\ 0 & 2+3-1 \end{pmatrix} =$$

$$= \begin{pmatrix} 236 & 0 \\ 0 & 4 \end{pmatrix}.$$

13b. $p(R) = 2I - 3R + R^5$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} (-1)^5 & 5\cdot(-1)^4 \\ 0 & (-1)^5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+3+(-1)^5 & -3+5\cdot(-1)^4 \\ 0 & 2+3+(-1)^5 \end{pmatrix} = \begin{pmatrix} 2+3-1 & -3+5 \\ 0 & 2+3-1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$$

13c. $p(S) = 2I - 3S + S^5$

$$S^5 = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}^5 = \left[\begin{pmatrix} -2/3 & 1 \\ 0 & -2/3 \end{pmatrix} (-3) \right]^5 = \begin{pmatrix} (-2/3)^5 & 5(-2/3)^4 \\ 0 & (-2/3)^5 \end{pmatrix} (-3)^5$$

$$= \begin{pmatrix} 2^5 & 5(+2)^4(-3) \\ 0 & 2^5 \end{pmatrix} = \begin{pmatrix} 32 & -240 \\ 0 & 32 \end{pmatrix}$$

$$p(S) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -6 & 9 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} 32 & -240 \\ 0 & 32 \end{pmatrix} = \begin{pmatrix} 28 & -231 \\ 0 & 28 \end{pmatrix}$$

14. Evaluate the following complex numbers and express your answer in Euler's form. You should take the argument between 0 and 2π.

(a) $z_1 = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}} \cdot \dfrac{1+i\sqrt{3}}{1+i\sqrt{3}} = \dfrac{1+2i\sqrt{3}-3}{4} = \dfrac{-2+2i\sqrt{3}}{4}$

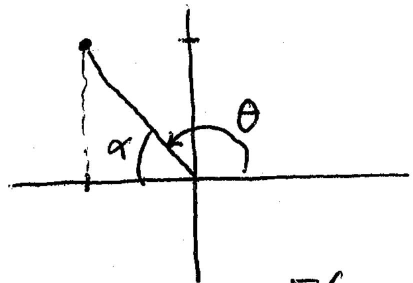
$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$= 1 \cdot e^{i \cdot 2\pi/3}$

$r^2 = \frac{1}{4} + \frac{3}{4}$

$r^2 = 1$

$r = 1$



$\tan\alpha = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\alpha = \pi/3$

$\theta = 2\pi/3$

(b) $z_2 = \left(\dfrac{1}{13} + \dfrac{5}{13}i\right) \cdot (3+2i)$

$= \frac{3}{13} + \frac{15}{13}i + \frac{2}{13}i - \frac{10}{13}$

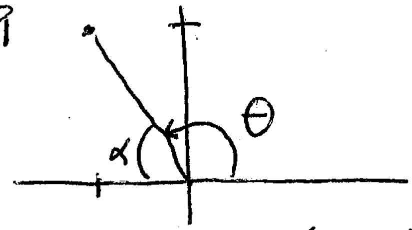
$= -\frac{7}{13} + \frac{17}{13}i$

$= \sqrt{2}\, e^{1.96 i}$

$r^2 = \dfrac{49}{169} + \dfrac{289}{169}$

$= 2$

$r = \sqrt{2}$



$\tan\alpha = \dfrac{17/13}{7/13} = \dfrac{17}{7}$

$\alpha = \tan^{-1}(17/7)$

$\theta = \pi - \alpha$

$= 1.96$

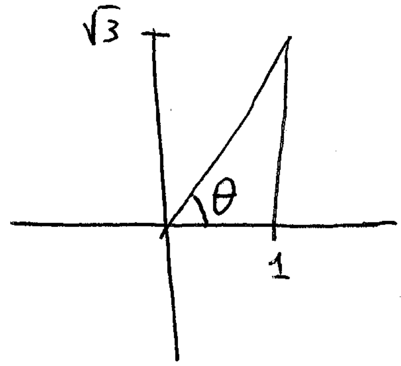
(c) $z_3 = (1+i\sqrt{3})^{32} = \left(2 e^{i \cdot \pi/3}\right)^{32}$

$= 2^{32} \cdot e^{i \cdot \frac{32\pi}{3}}$

$= 2^{32} \cdot e^{i \cdot 2\pi/3}$

$r^2 = 1+3$

$r = 2$



$\frac{32}{3}\pi$

$= (10 + \frac{2}{3})\pi$

$\sim 2\pi/3$

$\tan\theta = \sqrt{3}$

$\theta = \pi/3$

15. A study of a hermaphrodite organism concluded that there are four growth stages summarized in the table below

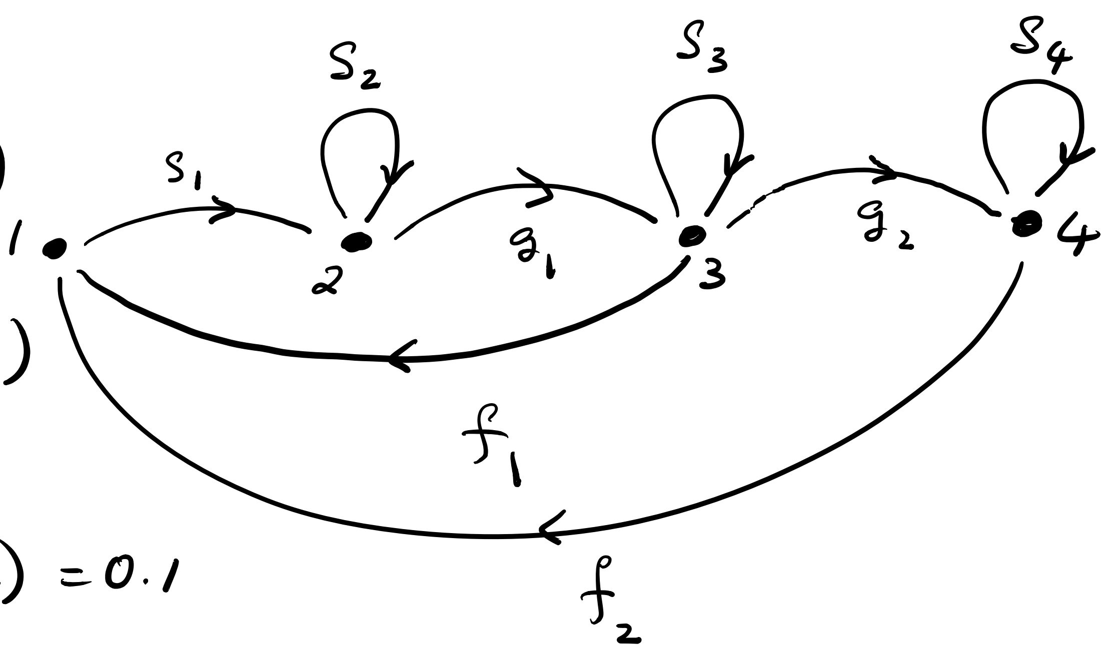
Stage	Description	Stage Duration (Years)	Annual Survivor Rates	Annual Fecundity
1	Eggs/Hatchlings	1	0.6	0
2	Juveniles	5	0.4	0
3	Young Adults	8	0.5	0
4	Adults	> 15	0.8	100

0.1

0.2

It is also known that **each year** 10% of those in Stage 2 will reach Stage 3, and 20% of those in Stage 3 will reach Stage 4. Answer the following questions assuming that census are taken right after nesting (eggs laying).

(a) Draw the lifecycle graph using the rates above. You should indicate the weights of the directed edges.



$s_1 = 0.6$

$g_1 = (0.4)(0.1)$

$= 0.04$

$S_2 = (0.4)(0.9)$

$= 0.36$

$g_2 = 0.5(0.2) = 0.1$

$S_3 = 0.5(0.8) = 0.4$; $S_4 = 0.8$

$f_1 = 100(g_2) = 100(0.1) = 10$

$f_2 = 100(S_4) = 100(0.8) = 80.$

(b) If $x_1(n)$, $x_2(n)$, $x_3(n)$, and $x_4(n)$ are the number of the organism of stages 1, 2, 3, and 4 respectively in the nth year, write down the system of difference equations that describes the growth of the population of the organism according to its growth stages.

$$x_1(n) = 10\,x_3(n-1) + 80\,x_4(n-1)$$

$$x_2(n) = 0.6\,x_1(n-1) + 0.36\,x_2(n-1)$$

$$x_3(n) = 0.04\,x_2(n-1) + 0.4\,x_3(n-1)$$

$$x_4(n) = 0.1\,x_3(n-1) + 0.8\,x_4(n-1)$$

(c) Write down the Lefkovitch matrix L for the population model above.

$$\vec{X}(n) = \begin{bmatrix} 0 & 0 & 10 & 80 \\ 0.6 & 0.36 & 0 & 0 \\ 0 & 0.04 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.8 \end{bmatrix} \vec{X}(n-1)$$

So $$L = \begin{bmatrix} 0 & 0 & 10 & 80 \\ 0.6 & 0.36 & 0 & 0 \\ 0 & 0.04 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.8 \end{bmatrix}$$

16. The MatLab output for the Lefkovitch matrix M given in the attached page is for an animal having five growth stages: Eggs, Juveniles, Sub-Adult A, Sub-Adult B, and Adult.

(a) Estimate the continuous grow rate of the population after a long time. Is the population increasing or decreasing after a long time?

Dominant eigenvalue $\lambda = 0.932$

$e^{k} = 0.932$

Continuous grow rate = $\ln(0.932)$

Circle One: Population Increases **Population Decreases** since $0.932 < 1$

(b) Find the stable population vector for the population. Your entries should be round to **four decimal places**.

$S = 1.2714 + 1.0551 + 0.0779 + 0.0094 + 0.01$
$= 2.4238.$

$$\text{stable population vector} = \frac{1}{S}\begin{bmatrix} 1.2714 \\ 1.0551 \\ 0.0779 \\ 0.0094 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.5245 \\ 0.4353 \\ 0.0321 \\ 0.0039 \\ 0.0041 \end{bmatrix}$$

(c) What is the population distribution in **percentages** after a long time? Give your answer round to **two decimal places**.

Eggs = 52.45 %

Juvenile stage = 43.53 %

Sub-adult A stage = 3.21 %

Sub-adult B stage = 0.39 %

Adult stage = 0.41 %

17. The MatLab information for a matrix A is given in the attached page. Answer the following question about A

(a) Write down the canonical form C of A with all real entries.

$$C = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(b) Find a real matrix P with all integer entries such that $A = PCP^{-1}$. Your entries for P should have **no unnecessary common factors.**

$$P = \begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & -5 & 0 & 0 \\ 3 & -1 & 0 & -2 \end{bmatrix}$$