Math 20480 Exam 03 Review Name ________________________________

1. Suppose a 3×3 matrix A has eigenvalues 2, 3, and 3. Let J be the (real) canonical form associated to A. In each of the following cases, write down e^J.

1a. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v} + n \cdot \vec{w}$ where k, m, and n are real constants.

1b. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v}$ where k, and m are real constants.

2. Suppose a 3×3 matrix A has eigenvalues -1, $3 \pm 4i$. Write down the (real) canonical form J associated to A. There are several possible answer depending on your construction, write down all of them. Also give the corresponding e^J.

3. Solve the system of equations:

$$\begin{aligned} x' &= 2x \\ y' &= -5x + 5z \\ z' &= -2x - y + 4z \end{aligned}$$

4.

$$M = \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

4a. Find the canonical form J associated to M.

4b. Let P be a matrix such that $M = PJP^{-1}$. Find exactly (every entry of) the matrices S_1 and S_2 as defined below. (Hint: It is not necessary that you know what P is)

4b. (i) $T(M) = PS_1P^{-1}$ where $T(x)$ is the polynomial $3 - x^2 + 2x^5$.

4b. (ii) $e^M = PS_2P^{-1}$

5. Evaluate each of the matrix exponentials. Here $Exp(A) = e^A$.

$$Exp\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix} \stackrel{?}{=}$$

$$Exp\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \stackrel{?}{=}$$

$$Exp\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 3 & -2 \end{pmatrix} \stackrel{?}{=}$$

6.

$$M = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 5 & -6 \\ 2 & 2 & -3 \end{pmatrix}$$

6a. The characteristic polynomial of M is $\lambda^3 - 2\lambda^2 - \lambda + 2$. Find all eigenvalues of M.

6b. Can M be diagonalized? If so write down the diagonal matrix J associated to M. If not, write down a more general canonical form J associated to M.

6c. Solve the system of differential equations:

$$\begin{array}{rcrcrcr} x' & = & & - & 2\,y & + & 2\,z \\ y' & = & 3\,x & + & 5\,y & - & 6\,z \\ z' & = & 2\,x & + & 2\,y & - & 3\,z \end{array}$$

7.

$$Q = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -3 & 5 \\ -4 & -5 & 7 \end{pmatrix}$$

The eigenvalues and their associate eigenvectors of Q are given below

$$\lambda_1 = 2 : \qquad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \qquad\qquad \lambda_2 = -1 + i : \qquad \vec{v} = \begin{pmatrix} 5s \\ (-1-2i)s \\ (2-i)s \end{pmatrix}$$

Solve the system of differential equations:

$$\begin{array}{rcrcrcr} x' & = & -4\,x & - & 5\,y & + & 5\,z \\ y' & = & -2\,x & - & 3\,y & + & 5\,z \\ z' & = & -4\,x & - & 5\,y & + & 7\,z \end{array}$$

8. The eigenvalues of $B = \begin{pmatrix} 5 & 4 & -4 \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ is λ =3, 3, and 3. Moreover the formulas of the eigenvectors $\vec{u}$ and generalized eigenvector $\vec{w}$ of B showing the JOrdan chain relation are given below

$$\vec{u} = \begin{pmatrix} -2r + 2s \\ r \\ s \end{pmatrix} \qquad\qquad \vec{w} = \begin{pmatrix} s - 2t + 2w \\ t \\ w \end{pmatrix}$$

where r, s, t and w are any real number.

Solve the system of differential equations with given initial conditions

$$\begin{array}{rcrcrcrl} x' & = & 5\,x & + & 4\,y & - & 4\,z & \quad x(0) = 1 \\ y' & = & & & 3\,y & & & \quad y(0) = -1 \\ z' & = & x & + & 2\,y & + & z & \quad z(0) = -2 \end{array}$$

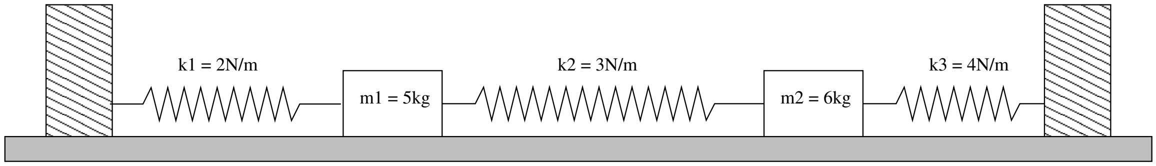
9. Solve the system of differential equations with given initial conditions

$$\begin{aligned} x'(t) &= \,x(t) - \,y(t) \qquad & x(0) &= 1 \\ y'(t) &= 2\,x(t) + 3\,y(t) \qquad & y(0) &= 2 \end{aligned}$$

10. Solve the system of differential equations with given initial conditions

$$\begin{aligned} x'(t) &= -4\,x(t) + 6\,y(t) \qquad & x(0) &= -1 \\ y'(t) &= -3\,x(t) + 5\,y(t) \qquad & y(0) &= 1 \end{aligned}$$

11.



A (undamped) double spring system is set up as shown below. The spring constants and masses are as labelled. The mass m_1 is displaced to the left $1/2$ meters and the mass m_2 is displaced to the right $1/3$ meters. Then m_1 and m_2 are released from rest. Write down the equations of motion that govern the resulting motion.

11a. Write down the equations of motion for the system above

11b. Rewrite your equations in Q11(a) as a linear first order system of equations.

12. The MatLab output for the Lefkovitch matrix M given in the attached page is for an animal having five growth stages: Eggs, Juveniles, Sub-Adult A, Sub-Adult B, and Adult.

(a) Estimate the continuous grow rate of the population after a long time. Is the population increasing or decreasing after a long time?

Continuous grow rate = ________________________

Circle One: Population Increases Population Decreases

(b) Find the stable population vector for the population. Your entries should be round to **four decimal places**.

(c) What is the population distribution in **percentages** after a long time? Give your answer round to **two decimal places**.

Eggs = ________________________

Juvenile stage = ________________________

Sub-adult A stage = ________________________

Sub-adult B stage = ________________________

Adult stage = ________________________

13. Solve the following system of equations:

$$\begin{aligned} x'(t) &= x(t) + y(t) + 4; \qquad & x(0) &= 1 \\ y'(t) &= -x(t) + 3\,y(t) + 8; \qquad & y(0) &= 1 \end{aligned}$$

Math 20480 – Matrix Formulas

You should know all these formulas for the exam.

Powers of 2×2 Canonical Forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}; \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^n = \begin{pmatrix} R\cos(\theta) & R\sin(\theta) \\ -R\sin(\theta) & R\cos(\theta) \end{pmatrix}^n = \begin{pmatrix} R^n\cos(n\theta) & R^n\sin(n\theta) \\ -R^n\sin(n\theta) & R^n\cos(n\theta) \end{pmatrix}$$

Powers of Larger Jordan Matrices

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \qquad \text{Here } \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} \\ 0 & \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & 0 & \lambda^n \end{pmatrix} \qquad \text{Here } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Exponential of 2×2 Canonical Forms

$$exp\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}; \qquad exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & 0 \\ 0 & \mu t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix};$$

$$exp\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Here $exp(A) = e^A$ where A is any square matrix.

Matrix Exponential Formulas (Real Eigenvalue Case).

$$exp\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} = \begin{pmatrix}e^{\lambda} & 0\\ 0 & e^{\mu}\end{pmatrix}\ ; \qquad exp\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} = \begin{pmatrix}e^{\lambda} & e^{\lambda}\\ 0 & e^{\lambda}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & 0\\ 0 & \mu t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{\mu t}\end{pmatrix};$$

$$exp\left[\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t\\ 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}$$

$$exp\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} = \begin{pmatrix}e^{\lambda_1} & 0 & 0\\ 0 & e^{\lambda_2} & 0\\ 0 & 0 & e^{\lambda_3}\end{pmatrix}\ ; \qquad exp\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix} = \begin{pmatrix}e^{\lambda} & e^{\lambda} & e^{\lambda}\\ 0 & e^{\lambda} & e^{\lambda}\\ 0 & 0 & e^{\lambda}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\right]\cdot t = exp\begin{pmatrix}\lambda_1 t & 0 & 0\\ 0 & \lambda_2 t & 0\\ 0 & 0 & \lambda_3 t\end{pmatrix} = \begin{pmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & e^{\lambda_2 t} & 0\\ 0 & 0 & e^{\lambda_3 t}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t & 0\\ 0 & \lambda t & t\\ 0 & 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}\\ 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & e^{\lambda t}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t & 0 & 0\\ 0 & \lambda t & t & 0\\ 0 & 0 & \lambda t & t\\ 0 & 0 & 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t} & t^3 e^{\lambda t}\\ 0 & e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}\\ 0 & 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & 0 & e^{\lambda t}\end{pmatrix}$$

Matrix Exponential Formulas (Complex Eigenvalue Case).

$$exp\begin{pmatrix}a & b\\ -b & a\end{pmatrix} = \begin{pmatrix}e^a\cos(b) & e^a\sin(b)\\ -e^a\sin(b) & e^a\cos(b)\end{pmatrix}$$

$$exp\left[\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}at & bt\\ -bt & at\end{pmatrix} = \begin{pmatrix}e^{at}\cos(bt) & e^{at}\sin(bt)\\ -e^{at}\sin(bt) & e^{at}\cos(bt)\end{pmatrix}$$

Remark 1. Let J be the canonical form of square matrix A and non-singular P such that $A = PJP^{-1}$. Then $e^A = P\cdot e^J\cdot P^{-1}$.

Remark 2. If $D = diag(A_1, A_2, \dots, A_k)$ where A_i is a square matrix of size n_i so D is a square matrix of size $N = \sum n_i$. Then we have:

$$e^D = diag(e^{A_1}, e^{A_2}, \dots, e^{A_k}).$$

MatLab Output for Lefkovitch Matrix M in Question 12

M =

0	0	2.2500	39.2000	64.0000
0.4000	0.4500	0	0	0
0	0.0500	0.2550	0	0
0	0	0.0450	0.5600	0
0	0	0	0.1400	0.8000

```
>> [P, J] = jordan(M)
```

P =

4.0800 + 0.0000i	-1.0577 - 0.9203i	-1.0577 + 0.9203i	-0.0094 + 0.0000i	1.2714 + 0.0000i
-3.6267 + 0.0000i	1.7498 - 0.4864i	1.7498 + 0.4864i	-0.0306 + 0.0000i	1.0551 + 0.0000i
0.7111 + 0.0000i	0.1259 + 0.3274i	0.1259 - 0.3274i	-0.0048 + 0.0000i	0.0779 + 0.0000i
-0.0571 + 0.0000i	-0.0372 - 0.0184i	-0.0372 + 0.0184i	-0.0162 + 0.0000i	0.0094 + 0.0000i
0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i	0.0100 + 0.0000i

J =

0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.2798 - 0.2577i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.2798 + 0.2577i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.5734 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i	0.9320 + 0.0000i

Math 20480 Exam 03 Review Name ______________________

1. Suppose a 3×3 matrix A has eigenvalues 2, 3, and 3. Let J be the (real) canonical form associated to A. In each of the following cases, write down e^J.

1a. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v} + n \cdot \vec{w}$ where k, m, and n are real constants.

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow e^J = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix}$$

1b. The eigenvectors for $\lambda = 2$ has the form $k \cdot \vec{u}$ and the eigenvectors for $\lambda = 3$ has the form $m \cdot \vec{v}$ where k, and m are real constants. Need generalized eigenvector for $\lambda = 3$

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow e^J = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & e^3 \\ 0 & 0 & e^3 \end{pmatrix}$$

2. Suppose a 3×3 matrix A has eigenvalues -1, $3 \pm 4i$. Write down the (real) canonical form J associated to A. There are several possible answer depending on your construction, write down all of them. Also give the corresponding e^J.

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix} ; \begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$e^J = \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^3\cos 4 & e^3 \sin 4 \\ 0 & -e^3 \sin 4 & e^3 \cos 4 \end{pmatrix}, \begin{pmatrix} e^3\cos 4 & e^3 \sin 4 & 0 \\ -e^3 \sin 4 & e^3 \cos 4 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix}$$

$$\begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^3\cos 4 & -e^3 \sin 4 \\ 0 & e^3 \sin 4 & e^3 \cos 4 \end{pmatrix}, \begin{pmatrix} e^3\cos 4 & -e^3 \sin 4 & 0 \\ e^3 \sin 4 & e^3 \cos 4 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix}$$

3. Solve the system of equations:

$$\begin{aligned} x' &= 2x \\ y' &= -5x + 5z \\ z' &= -2x - y + 4z \end{aligned}$$

$$\vec{X}'(t) = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ -5 & 0 & 5 \\ -2 & -1 & 4 \end{pmatrix}}_{A} \vec{X}(t)$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -5 & -\lambda & 5 \\ -2 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 5) = 0$$

$$\lambda = 2 \; ; \quad \lambda^2 - 4\lambda + 5 = 0 \Rightarrow (\lambda-2)^2 + 1 = 0$$
$$\Rightarrow \lambda = 2 \pm i$$

$\underline{\lambda = 2}$: $(A - 2I)\vec{u} = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -5 & -2 & 5 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right)$

$$r_2 - 2r_3 \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right) \Rightarrow \begin{cases} -u_1 + u_3 = 0 \Rightarrow u_1 = u_3 \\ -2u_1 - u_2 + 2u_3 = -u_2 = 0 \end{cases}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ 0 \\ u_1 \end{pmatrix}$$

$\underline{\lambda = 2 + i}$: $(A - (2+i)I)\vec{v} = 0$; $\left(\begin{array}{ccc|c} -i & 0 & 0 & 0 \\ -5 & -(2+i) & 5 & 0 \\ -2 & -1 & 2-i & 0 \end{array}\right)$

$-iv_1 = 0 \Rightarrow v_1 = 0$

$-(2+i)v_2 + 5v_3 = 0$

$-v_2 + (2-i)v_3 = 0$

$\hookrightarrow v_2 = (2-i)v_3$

$$\vec{v} = \begin{pmatrix} 0 \\ (2-i)v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2v_3 \\ v_3 \end{pmatrix} + i\begin{pmatrix} 0 \\ -v_3 \\ 0 \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} ; \quad J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\vec{X}(t) = e^{A \cdot t}\, \vec{X}(0) = P e^{J \cdot t} \cdot P^{-1} \vec{X}(0)$$

where $A = PJP^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t}\cos t & e^{2t}\sin t \\ 0 & -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t}\cos t + c_3 e^{2t}\sin t \\ -c_2 e^{2t}\sin t + c_3 e^{2t}\cos t \end{bmatrix}$$

$$x(t) = c_1 e^{2t}$$

$$y(t) = (2c_2 - c_3)e^{2t}\cos t + (2c_3 + c_2)e^{2t}\sin t$$

$$z(t) = c_1 e^{2t} + c_2 e^{2t}\cos t + c_3 e^{2t}\sin t.$$

4.

$$M = \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

4a. Find the canonical form J associated to M.

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$|M-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & -2 & 0 \\ 0 & -1-\lambda & 1 & 0 \\ 0 & -2 & 1-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -1-\lambda & 1 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda)(-(1-\lambda^2)+2) = (2-\lambda)(-1-\lambda)(\lambda^2+1) = 0$$

$$\lambda = -1,\ 2,\ \pm i$$

4b. Let P be a matrix such that $M = PJP^{-1}$. Find exactly (every entry of) the matrices S_1 and S_2 as defined below. (Hint: It is not necessary that you know what P is)

4b. (i) $T(M) = PS_1P^{-1}$ where $T(x)$ is the polynomial $3 - x^2 + 2x^5$.

$$T(M) = 3I_4 - M^2 + 2M^5 = 3P\cdot P^{-1} - PJ^2P^{-1} + 2PJ^5P^{-1}$$

$$= P(3I_4 - J^2 + 2J^5)P^{-1}$$

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \cos\pi/2 & \sin\pi/2 \\ 0 & 0 & -\sin\pi/2 & \cos\pi/2 \end{bmatrix}$$

So $S_1 = 3I_4 - J^2 + 2J^5$

$-J^2 = diag(-1, -4, 1, 1)$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 63 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$2J^5 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

4b. (ii) $e^M = PS_2P^{-1}$

$$= e^{PJP^{-1}} = Pe^JP^{-1} = \begin{bmatrix} e^{-1} & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & \cos 1 & \sin 1 \\ 0 & 0 & -\sin 1 & \cos 1 \end{bmatrix}$$

5. Evaluate each of the matrix exponentials. Here $Exp(A) = e^A$.

$$Exp\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} e^3 & e^3 & 0 & 0 \\ 0 & e^3 & 0 & 0 \\ 0 & 0 & e^{-3} & e^{-3} \\ 0 & 0 & 0 & e^{-3} \end{bmatrix}$$

$$Exp\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \stackrel{?}{=} Exp\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & \frac{3}{2}\cdot 2 & 2 \\ 0 & 0 & 0 & \frac{3}{2}\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} e^3 & e^3 & 0 & 0 \\ 0 & e^3 & 0 & 0 \\ 0 & 0 & e^3 & 2\cdot e^3 \\ 0 & 0 & 0 & e^3 \end{bmatrix}$$

$$exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$Exp\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 3 & -2 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} e^2 & 0 & 0 & 0 \\ 0 & e^{-2} & 0 & 0 \\ 0 & 0 & e^{-2}\cos 3 & -e^{-2}\sin 3 \\ 0 & 0 & e^{-2}\sin 3 & e^{-2}\cos 3 \end{bmatrix}$$

6.

$$M = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 5 & -6 \\ 2 & 2 & -3 \end{pmatrix}$$

6a. The characteristic polynomial of M is $\lambda^3 - 2\lambda^2 - \lambda + 2$. Find all eigenvalues of M.

$\lambda = 1: \quad \cancel{1} \ \cancel{-2} \ \cancel{-1} + \cancel{2} = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & 2 \\ & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Delta(\lambda) = (\lambda-1)(\lambda^2-\lambda-2)$

$= (\lambda-1)(\lambda+1)(\lambda-2)$

$\lambda = 1, -1, 2.$

6b. Can M be diagonalized? If so write down the diagonal matrix J associated to M. If not, write down a more general canonical form J associated to M.

Yes, there 3 distinct eigenvalues.

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

6c. Solve the system of differential equations:

$$\begin{aligned} x' &= \quad\;\; - 2y + 2z \\ y' &= 3x + 5y - 6z \\ z' &= 2x + 2y - 3z \end{aligned}$$

$$\vec{X}' = \underbrace{\begin{bmatrix} 0 & -2 & 2 \\ 3 & 5 & -6 \\ 2 & 2 & -3 \end{bmatrix}}_{M} \vec{X} \Rightarrow \vec{X} = e^{M \cdot t} \cdot \vec{X}(0)$$

$\underline{\lambda = -1}$: $(M+I)\vec{u} = 0$ $\quad \left[\begin{array}{ccc|c} +1 & -2 & 2 & 0 \\ 3 & 6 & -6 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 1 & 2 & -2 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & 2 & \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \end{array}\right] \Rightarrow \begin{aligned} u_1 &= 0 \\ -2u_2 + 2u_3 &= 0 \\ u_3 &= u_2 \end{aligned}$$

$$\vec{u} = \begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}.$$

$\underline{\lambda = 1}$: $(M - I)\vec{v} = 0 \Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 3 & 4 & -6 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 3 & 4 & -6 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \underset{r_2 + 3r_1}{\overset{r_1 + r_3}{\Longrightarrow}} \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$v_2 = 0 \quad ; \quad v_1 - 2v_3 = 0 \Rightarrow v_1 = 2v_3$$

$$\vec{v} = \begin{pmatrix} 2v_3 \\ 0 \\ v_3 \end{pmatrix}$$

$\underline{\lambda = 2}$: $(M - 2I)\vec{w} = 0 \Rightarrow \left[\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ 3 & 3 & -6 & 0 \\ 2 & 2 & -5 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \Rightarrow \begin{aligned} w_3 &= 0 \\ w_1 + w_2 &= 0 \\ w_2 &= -w_1 \end{aligned}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ -w_1 \\ 0 \end{pmatrix}$$

$$\underline{P} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} ; \quad J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\vec{X} = P e^{J \cdot t} \cdot P^{-1} \vec{X}(0)$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{t} \\ c_3 e^{2t} \end{bmatrix}$$

$$x(t) = 2c_2 e^{t} + c_3 e^{2t}$$

$$y(t) = c_1 e^{-t} - c_3 e^{2t}$$

$$z(t) = c_1 e^{-t} + c_2 e^{t}$$

7.

$$Q = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -3 & 5 \\ -4 & -5 & 7 \end{pmatrix}$$

The eigenvalues and their associate eigenvectors of Q are given below

$$\lambda_1 = 2 : \qquad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \qquad\qquad \lambda_2 = -1 + i : \qquad \vec{v} = \begin{pmatrix} 5s \\ (-1-2i)s \\ (2-i)s \end{pmatrix}$$

Solve the system of differential equations:

$$\begin{aligned} x' &= -4x - 5y + 5z \\ y' &= -2x - 3y + 5z \\ z' &= -4x - 5y + 7z \end{aligned}$$

$$\vec{X}'(t) = \underbrace{\begin{bmatrix} -4 & -5 & 5 \\ -2 & -3 & 5 \\ -4 & -5 & 7 \end{bmatrix}}_{Q} \vec{X}(t) \Longrightarrow \vec{X}(t) = e^{Q\cdot t}\vec{X}(0)$$

$$Q = PJP^{-1} \text{ where } P = \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\vec{X} = P e^{J\cdot t} P^{-1} \vec{X}(0)$$

$$= \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t}\cos t & e^{-t}\sin t \\ 0 & -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 \\ 1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{-t}\cos t + C_3 e^{-t}\sin t \\ -C_2 e^{-t}\sin t + C_3 e^{-t}\cos t \end{bmatrix}$$

$$= \begin{bmatrix} 5C_2 e^{-t}\cos t + 5C_3 e^{-t}\sin t \\ C_1 e^{2t} - (C_2 + 2C_3)e^{-t}\cos t + (2C_2 - C_3)e^{-t}\sin t \\ C_1 e^{2t} + (2C_2 - C_3)e^{-t}\cos t + (C_2 + 2C_3)e^{-t}\sin t \end{bmatrix}$$

8. The eigenvalues of $B = \begin{pmatrix} 5 & 4 & -4 \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ is λ =3, 3, and 3. Moreover the eigenvectors $\vec{u}$ and generalized eigenvector $\vec{w}$ of B are given below

$$\vec{u} = \begin{pmatrix} -2r + 2s \\ r \\ s \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} s + 2t \\ 0 \\ t \end{pmatrix}$$

where r, s and t are any real number.

Solve the system of differential equations with given initial conditions

$$\begin{aligned} x' &= 5x + 4y - 4z \qquad & x(0) &= 1 \\ y' &= 3y & y(0) &= -1 \\ z' &= x + 2y + z & z(0) &= -2 \end{aligned}$$

$$\vec{u} = r\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad ; \quad \vec{w} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Pick : $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $s = 1, t = 0$

$\vec{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $\quad r = 0$

$\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ $\quad r = 1, \; s = 0$

$$P = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \; ; \; J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\vec{X}' = B \cdot \vec{X} \Rightarrow \vec{X} = e^{B \cdot t} \cdot \vec{X}(0) = P e^{J \cdot t} \cdot P^{-1} \vec{X}(0)$$

$$\vec{X} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2c_1 + 2c_2 + c_3 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$c_1 = -1 \; ; \; c_2 = -2 \; ; \; c_3 = 1 + 2c_1 - 2c_2 = 1 - 2 + 4 = 3$$

8. The eigenvalues of $B = \begin{pmatrix} 5 & 4 & -4 \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ is λ =3, 3, and 3. Moreover the formulas of the eigenvectors $\vec{u}$ and generalized eigenvector $\vec{w}$ of B showing the JOrdan chain relation are given below

$$\vec{u} = \begin{pmatrix} -2r + 2s \\ r \\ s \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} s - 2t + 2w \\ t \\ w \end{pmatrix}$$

where r, s, t and w are any real number.

Solve the system of differential equations with given initial conditions

$$\begin{aligned} x' &= 5x + 4y - 4z \qquad & x(0) &= 1 \\ y' &= 3y \qquad & y(0) &= -1 \\ z' &= x + 2y + z \qquad & z(0) &= -2 \end{aligned}$$

$$\vec{u} = r\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \ ; \quad \vec{w} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Pick : $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow s = 1, t = 0 = w$

$\vec{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad r = 0$

$\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad r = 1, \ s = 0$

$$P = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ; \quad J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\vec{X}' = B\cdot\vec{X} \Rightarrow \vec{X} = e^{B\cdot t}\cdot\vec{X}(0) = P e^{J\cdot t}\cdot P^{-1}\vec{X}(0)$$

$$\vec{X} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2c_1 + 2c_2 + c_3 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$c_1 = -1 \ ; \ c_2 = -2 \ ; \ c_3 = 1 + 2c_1 - 2c_2 = 1 - 2 + 4 = 3$$

$$\vec{X} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -e^{3t} \\ -2e^{3t} + 3te^{3t} \\ 3e^{3t} \end{bmatrix}$$

$$x(t) = 2e^{3t} - 4e^{3t} + 6te^{3t} + 3e^{3t} = e^{3t} + 6te^{3t}$$

$$y(t) = -e^{3t}$$

$$z(t) = -2e^{3t} + 3te^{3t}.$$

Q9. $x' = x - y \quad ; \quad x(0) = 1$

$y' = 2x + 3y \quad ; \quad y(0) = 2$

$$\Rightarrow \vec{X}' = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}}_{A} \vec{X} \quad ; \quad \vec{X}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2$$

$$= 3 - 4\lambda + \lambda^2 + 2 = \lambda^2 - 4\lambda + 5 = \lambda^2 - 4\lambda + 4 + 1$$

$$= (\lambda - 2)^2 + 1 = 0 \Rightarrow \lambda = 2 \pm i$$

$$(A - (2+i)I)\vec{u} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right)$$

$$\Rightarrow -(1+i)u_1 - u_2 = 0 \Rightarrow u_2 = (-1-i)u_1$$

$$\Rightarrow \vec{u} = \begin{pmatrix} u_1 \\ (-1-i)u_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} u_1}_{\vec{p}} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix} u_1}_{\vec{q}}$$

$$\underline{P} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad ; \quad J = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \leftarrow \text{canonical form of } A.$$

$$\text{So } \vec{X}(t) = e^{At} \cdot \vec{X}(0) = P e^{Jt} \underbrace{P^{-1} \cdot \vec{X}(0)}_{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -e^{2t}\sin t & e^{2t}\cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\vec{X}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$c_1 = 1 \quad ; \quad -c_1 - c_2 = 2 \Rightarrow c_2 = -2 - c_1 = -3$$

$$\vec{X}(t) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -e^{2t}\sin t & e^{2t}\cos t \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t \\ -e^{2t}\sin t + 3e^{2t}\cos t \end{pmatrix}$$

$$x(t) = e^{2t}\cos t - 3e^{2t}\sin t$$

$$y(t) = -e^{2t}\cos t + 3e^{2t}\sin t + e^{2t}\sin t - 3e^{2t}\cos t$$

$$= -4e^{2t}\cos t + 4e^{2t}\sin t$$

9a

Q10. $x'(t) = -4x(t) + 6y(t)$; $x(0) = -1$

$y'(t) = -3x(t) + 5y(t)$; $y(0) = 1$

$$\vec{X}' = \underbrace{\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}}_{A} \vec{X} \quad ; \quad \vec{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) + 18$$

$$= \lambda^2 - 5\lambda + 4\lambda - 20 + 18 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$\Rightarrow \lambda = -1, 2$

$\underline{\lambda = -1}$: $(A + I)\vec{u} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -3 & 6 & 0 \\ -3 & 6 & 0 \end{array}\right)$

$\Rightarrow -3u_1 + 6u_2 = 0 \Rightarrow u_1 = 2u_2 \Rightarrow \vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_2$

$\underline{\lambda = 2}$: $(A - 2I)\vec{v} = 0 \Rightarrow \left(\begin{array}{cc|c} -6 & 6 & 0 \\ -3 & 3 & 0 \end{array}\right)$

$\Rightarrow -6v_1 + 6v_2 = 0 \Rightarrow v_1 = v_2 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1$

$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$; $J = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \leftarrow$ Canonical form of A

So $\vec{X}(t) = e^{At} \cdot \vec{X}(0) = P e^{Jt} \underbrace{P^{-1} \vec{X}(0)}_{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\vec{X}(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} 2c_1 + c_2 &= -1 \quad — ① \\ c_1 + c_2 &= 1 \quad — ② \end{aligned}\right\} \Rightarrow \underline{① - ②} : c_1 = -2$$

$$\Rightarrow c_2 = 1 - c_1 = 3$$

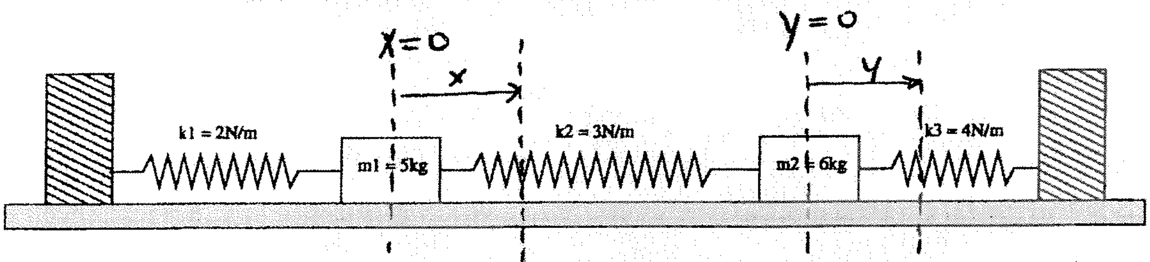
$$\vec{X}(t) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2e^{-t} \\ 3e^{2t} \end{pmatrix}$$

$$x(t) = -4e^{-t} + 3e^{2t}$$

$$y(t) = -2e^{-t} + 3e^{2t}$$

11.



A (undamped) double spring system is set up as shown below. The spring constants and masses are as labelled. The mass m_1 is displaced to the left 1/2 meters and the mass m_2 is displaced to the right 1/3 meters. Then m_1 and m_2 are released from rest. Write down the equations of motion that govern the resulting motion.

11a. Write down the equations of motion for the system above

$$m_1 \ddot{x} = -2x - 3x + 3y$$

$$m_1 \ddot{x} = -5x + 3y$$

$$m_2 \ddot{y} = +3x - 3y - 4y$$
$$= 3x - 7y$$

$$\ddot{x} = -x + \frac{3}{5}y$$

$$\ddot{y} = \frac{1}{2}x - \frac{7}{6}y$$

$$x(0) = -1/2$$

$$y(0) = 1/3$$

11b. Rewrite your equations in Q11(a) as a linear first order system of equations.

$$u = \dot{x} \Rightarrow \dot{u} = \ddot{x} = -x + \frac{3}{5}y$$

$$v = \dot{y} \Rightarrow \dot{v} = \ddot{y} = \frac{1}{2}x - \frac{7}{6}y$$

$$\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{u} = -x + \frac{3}{5}y \\ \dot{v} = \frac{1}{2}x - \frac{7}{6}y \end{cases}$$

$$x(0) = -1/2$$

$$y(0) = 1/3$$

$$u(0) = \dot{x}(0) = 0$$

$$v(0) = \dot{y}(0) = 0$$

at rest at t=0

12. The MatLab output for the Lefkovitch matrix M given in the attached page is for an animal having five growth stages: Eggs, Juveniles, Sub-Adult A, Sub-Adult B, and Adult.

(a) Estimate the continuous grow rate of the population after a long time. Is the population increasing or decreasing after a long time?

Dominant eigenvalue $\lambda = 0.932$

$e^k = 0.932$

Continuous grow rate = $\ln(0.932)$

Circle One: Population Increases Population Decreases since $0.932 < 1$

(b) Find the stable population vector for the population. Your entries should be round to **four decimal places**.

$$S = 1.2714 + 1.0581 + 0.0779 + 0.0094 + 0.01$$
$$= 2.4238.$$

$$\text{stable population vector} = \frac{1}{S}\begin{bmatrix} 1.2714 \\ 1.0551 \\ 0.0779 \\ 0.0094 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.5245 \\ 0.4353 \\ 0.0321 \\ 0.0039 \\ 0.0041 \end{bmatrix}$$

(c) What is the population distribution in **percentages** after a long time? Give your answer round to **two decimal places.**

Eggs = 52.45 %

Juvenile stage = 43.53 %

Sub-adult A stage = 3.21 %

Sub-adult B stage = 0.39 %

Adult stage = 0.41 %

13. Solve the following system of equations:

$$
\begin{aligned}
x'(t) &= \quad x(t) + \ \ y(t) \ + 4; \qquad x(0) = 1\\
y'(t) &= -x(t) + 3\,y(t) \ + 8; \qquad y(0) = 1
\end{aligned}
$$

$$\vec{X}'(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}}_{A} \vec{X}(t) + \underbrace{\begin{pmatrix} 4 \\ 8 \end{pmatrix}}_{\vec{B}} = A\vec{X} + AA^{-1}\vec{B}$$

$$= A\left(\vec{X} + A^{-1}\vec{B}\right)$$

$$A^{-1}\vec{B} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Let $\vec{Y} = \vec{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

So $\vec{Y}' = A\vec{Y} \Rightarrow \vec{Y}(t) = e^{At} \cdot \vec{Y}(0)$

Eigenvalues and Eigenspaces of A.

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$$

$\Rightarrow \lambda = 2, 2$

Eigenspace for $\lambda = 2$

$$(A - \lambda I)\vec{u} = \vec{0} \Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -u_1 + u_2 = 0$$

$$\Rightarrow u_2 = u_1$$

$\vec{u} = \begin{bmatrix} u_1 \\ u_1 \end{bmatrix}$; $u_1 \in \mathbb{R}$

Generalized eigenspace for $\lambda = 2$

$$(A - \lambda I)\vec{v} = \vec{u} \Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} \Rightarrow -v_1 + v_2 = u_1$$

$$\Rightarrow v_2 = v_1 + u_1$$

$\vec{v} = \begin{bmatrix} v_1 \\ v_1 + u_1 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $u_1, v_1 \in \mathbb{R}$

Pick $u_1 = 1$, $v_1 = 0$ so $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

So $A = PJP^{-1} \Rightarrow At = P \cdot Jt \cdot P^{-1}$

$$e^{At} = P \cdot e^{J \cdot t} \cdot P^{-1} \quad ; \quad Jt = \begin{pmatrix} 2t & t \\ 0 & 2t \end{pmatrix}$$

$$\vec{Y}(t) = e^{At} \cdot \vec{Y}(0) = P \cdot e^{Jt} \cdot P^{-1} \vec{Y}(0)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ (c_1 + c_2) e^{2t} + c_2 t e^{2t} \end{pmatrix}$$

$$\vec{Y} = \vec{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \vec{X} = \vec{Y} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x(t) = c_1 e^{2t} + c_2 t e^{2t} - 1 \\ y(t) = (c_1 + c_2) e^{2t} + c_2 t e^{2t} - 3 \end{cases}$$

$$x(0) = 1 = c_1 - 1 \Rightarrow c_1 = 2$$

$$y(0) = 1 = c_1 + c_2 - 3 \Rightarrow c_2 = -c_1 + 4 = 2$$

$$x(t) = 2e^{2t} + 2te^{2t} - 1$$

$$y(t) = 4e^{2t} + 2te^{2t} - 3$$

Math 20480 – Matrix Formulas

You should know all these formulas for the exam.

Powers of 2×2 Canonical Forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}; \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^n = \begin{pmatrix} R\cos(\theta) & R\sin(\theta) \\ -R\sin(\theta) & R\cos(\theta) \end{pmatrix}^n = \begin{pmatrix} R^n\cos(n\theta) & R^n\sin(n\theta) \\ -R^n\sin(n\theta) & R^n\cos(n\theta) \end{pmatrix}$$

Powers of Larger Jordan Matrices

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \qquad \text{Here } \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} \\ 0 & \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & 0 & \lambda^n \end{pmatrix} \qquad \text{Here } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Exponential of 2×2 Canonical Forms

$$exp\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}; \qquad exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & 0 \\ 0 & \mu t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix};$$

$$exp\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Here $exp(A) = e^A$ where A is any square matrix.

Matrix Exponential Formulas (Real Eigenvalue Case).

$$exp\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}; \qquad exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & 0 \\ 0 & \mu t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix};$$

$$exp\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$exp\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix}; \qquad exp\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} e^{\lambda} & e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} & e^{\lambda} \\ 0 & 0 & e^{\lambda} \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\right] \cdot t = exp\begin{pmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & t & 0 \\ 0 & \lambda t & t \\ 0 & 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

$$exp\left[\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} \lambda t & t & 0 \\ 0 & \lambda t & t \\ 0 & 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

Matrix Exponential Formulas (Complex Eigenvalue Case).

$$exp\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} e^{a}\cos(b) & e^{a}\sin(b) \\ -e^{a}\sin(b) & e^{a}\cos(b) \end{pmatrix}$$

$$exp\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot t\right] = exp\begin{pmatrix} at & bt \\ -bt & at \end{pmatrix} = \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}$$

Remark 1. Let J be the canonical form of square matrix A and non-singular P such that $A = PJP^{-1}$. Then $e^A = P \cdot e^J \cdot P^{-1}$.

Remark 2. If $D = diag(A_1, A_2, \ldots, A_k)$ where A_i is a square matrix of size n_i so D is a square matrix of size $N = \sum n_i$. Then we have:

$$e^D = diag(e^{A_1}, e^{A_2}, \ldots, e^{A_k}).$$