**Math 20480 – Example Set 03A**

**Determinant of A Square Matrices.**

**Case:** $2 \times 2$ **Matrices.**

The determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{\hspace{5cm}}$$

For matrices with size larger than 2, we need to use cofactor expansion to obtain its value.

**Case:** $3 \times 3$ **Matrices.**

The determinant of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is $det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(Cofactor expansion along the 1st row)

$$= a_{11} \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} - a_{12} \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} + a_{13} \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$$

(Cofactor expansion along the 2nd row)

$$=$$

(Cofactor expansion along the 1st column)

$$=$$

(Cofactor expansion along the 2nd column)

$$=$$

**Example 1:**

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{vmatrix} \stackrel{?}{=}$$

**Case:** $4 \times 4$ **Matrices.**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

(Cofactor expansion along the 1st row)

$=$

(Cofactor expansion along the 2nd column)

$=$

**Example 2:**

$$\begin{vmatrix} 1 & -2 & -1 & 3 \\ -1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 2 \\ 3 & -1 & 2 & -3 \end{vmatrix} \stackrel{?}{=}$$

**Theorem 1:** Let $A$ be a square $n \times n$ matrix. Then the $A$ has an inverse if and only if $det(A) \neq 0$. We call a matrix $A$ with inverse a ________________________ matrix.

**Theorem 2:** Let $A$ be a square $n \times n$ matrix and $\vec{b}$ be any $n \times 1$ matrix. Then the equation

$$A\vec{x} = \vec{b}$$

has a ________________ solution for $\vec{x}$ if and only if ________________.

**Proof:**

**Remark:** If $A$ is **non-singular** then the only solution of $A\vec{x} = 0$ is $\vec{x} =$ ________.

**3.** Without solving explicitly, determine if the following systems of equations have a **unique** solution.

**a.**

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 8 \\ x & - & 2y & - & z & = & -1 \\ x & - & y & + & z & = & 4 \end{array}$$

**b.**

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 3 \\ -2x & & & + & z & = & -3 \\ -5x & + & y & + & z & = & -6 \end{array}$$

**Geometric Interpretation of $n$-tuples**

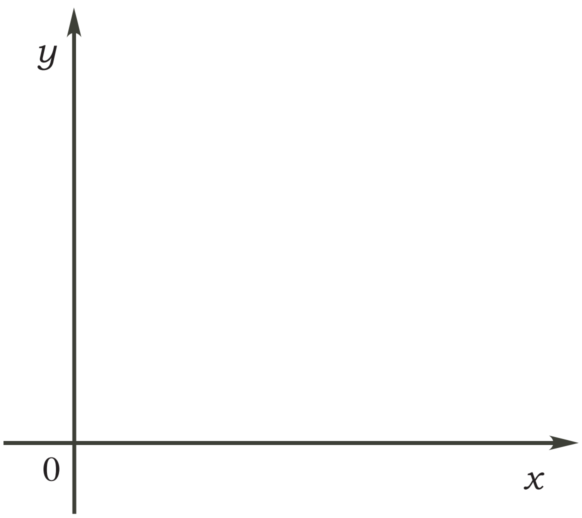
**Case:** 2 -tuples $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} x & y \end{pmatrix}$

There are two geometric interpretations of 2-tuples:

(A) ________________________ or (B) ________________________

**Position Vectors**

Each tuple $\begin{pmatrix} x \\ y \end{pmatrix}$ can be used to locate a ____________ on plane relative to the ____________.
Position vectors are always based at the ____________.




(1) Concept of direction from the origin

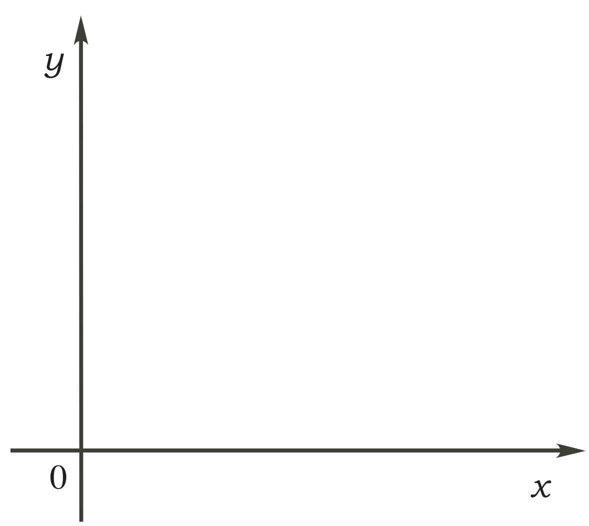
(2) Concept of distance

**Direction Vectors**

Each tuple $\begin{pmatrix} x \\ y \end{pmatrix}$ can also be used to indicate the a direction.

(1) Direction vectors are ________________________free.

(2) Parallel vectors are identified.



**The $xy-$plane as a Two Dimension Vector Space**

Interpret each tuple $\begin{pmatrix} x \\ y \end{pmatrix}$ as a matrix. Then we may write the sum:

We call this sum a ______________________________ of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We can interpret the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as ______________________________ for the two-dimensional coordinate space.

We call the set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ the ______________________________.

**Example 4:** Can the set $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ form a basis for the two-dimensional coordinate space? Explain. If yes, write the $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Illustrate your answer in a diagram.

**Basis for The Two Dimensional Space $\mathbb{R}^2$**

The two-dimensional coordinate space is often denoted by $\mathbb{R}^2$. We have seen that all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ can be ____________ written as a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\mathbb{R}^2 = \left\{ x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} : x, y \text{ are real numbers} \right\}$$
$$=$$

We observe that $\mathbb{R}^2$ is "built up" by ____________ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We need exact two vectors to span the space $\mathbb{R}^2$.

**No more:** What if we use three vectors to span: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$?

**No less:** What if only have one vector to span: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} a \\ b \end{pmatrix}$?

**Example 5:** Can any vector in $\mathbb{R}^2$ be uniquely written as a linear combination of $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$? Explain.

**Theorem 3:** Two vectors $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ forms a basis for $\mathbb{R}^2$ if and only if $\begin{vmatrix} a & c \\ b & d \end{vmatrix} \neq 0$.

**Theorem 4:** $n$ vectors $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_n$ forms a basis in $\mathbb{R}^n$ if and only if the determinant

$$|\vec{u}_1 : \vec{u}_2 : \cdots : \vec{u}_n| \neq 0.$$

**Example 6:** Which of the sets below form a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$

**Formal Definition of A Vector Space** A vector space $V$ over the real numbers is a set with two operations $+$ (addition) and $\cdot$ (scalar multiplication) with the following properties:

(1) $V$ is an non-empty set whose members are called ________________. Moreover, $V$ is ________________ under addition and scalar multiplications.

(2) **Transitivity:** $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for any members $\mathbf{u},\mathbf{v},\mathbf{w}$ of $V$.

(3) **Zero Vector:** There is a vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}$ for any $\mathbf{u}$ in $V$.

(4) **Negative Vector:** For each $\mathbf{u}$ in $V$, there is a vector in $V$, denoted $-\mathbf{u}$, called the negative of $\mathbf{u}$ such that

$$\mathbf{u}+(-\mathbf{u})=\mathbf{0}=(-\mathbf{u})+\mathbf{u}.$$

(5) **Commutative:** $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for any $\mathbf{u},\mathbf{v}$ in $V$.

(6) **Scalar multiplication 1:** For any real number (scalar) $k$, and any $\mathbf{u}, \mathbf{v}$ in $V$,

$$k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$$

(7) **Scalar multiplication 2:** For any real numbers (scalars) $a$ and $b$, and any $\mathbf{u}$ in $V$,

$$(a + b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$$

(8) **Scalar multiplication 3:** For any real numbers (scalars) $a$ and $b$, and any $\mathbf{u}$ in $V$,

$$(ab) \cdot \mathbf{u} = a \cdot (b \cdot \mathbf{u})$$

(9) **Unit Scalar:** There is a unit scalar, denoted 1, such that for any $\mathbf{u}$,

$$1 \cdot \mathbf{u} = \mathbf{u}.$$

**Examples of Vector Spaces over $\mathbb{R}$.**

(1) The $n$-dimensional real coordinate space $\mathbb{R}^n$ with the usual vector addition and scaling by real numbers. It standard basis is:

(2) The space of all real $2 \times 2$ matrices with the usual matrix addition and scaling by real numbers. It standard basis is:

(3) The space of all real $2 \times 3$ matrices with the usual matrix addition and scaling by real numbers. It standard basis is:

(4) The space of all quadratic polynomials with real coefficients with usual matrix addition and scaling by real numbers. It standard basis is:

(5) The solution space of $A\vec{x} = 0$ where $A$ is any $n \times n$ real matrix (System of homogeneous solution).

**Example 6:** Consider the system of linear homogeneous equations:

$$\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 0 \\ -2x & & & + & z & = & 0 \\ -5x & + & y & + & z & = & 0 \end{array}$$

Verify that the solutions of these equations forms a vector space. Find a basis for its solution space $V$.

(1) $V$ is an non-empty set closed under addition and scalar multiplications.

(2) **Transitivity:** $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for any members $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of $V$.

(3) **Zero Vector:** There is a vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u}$ for any $\mathbf{u}$ in $V$.

(4) **Negative Vector:** For each $\mathbf{u}$ in $V$, there is a vector in $V$, denoted $-\mathbf{u}$, called the negative of $\mathbf{u}$ such that

$$\mathbf{u}+(-\mathbf{u}) = \mathbf{0} = (-\mathbf{u})+\mathbf{u}.$$

(5) **Commutative:** $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for any $\mathbf{u}, \mathbf{v}$ in $V$.

(6) **Scalar multiplication 1:** For any real number (scalar) $k$, and any $\mathbf{u}, \mathbf{v}$ in $V$,

$$k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$$

(7) **Scalar multiplication 2:** For any real numbers (scalars) $a$ and $b$, and any $\mathbf{u}$ in $V$,

$$(a + b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$$

(8) **Scalar multiplication 3:** For any real numbers (scalars) $a$ and $b$, and any $\mathbf{u}$ in $V$,

$$(ab) \cdot \mathbf{u} = a \cdot (b \cdot \mathbf{u})$$

(9) **Unit Scalar:** There is a unit scalar, denoted 1, such that for any $\mathbf{u}$,

$$1 \cdot \mathbf{u} = \mathbf{u}.$$

To find a basis for the solution space we need to find the solution using Gaussian elimination.

**Linear Independence**

**Example 7:** Describe the (linear) span of the following set of vectors in $\mathbb{R}^3$?

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

**Definition (Linear Dependence:** A set of vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ is said to be linearly dependent if

there are scalars $a_1, a_2, ..., a_n$, ____________________________ , such that

Otherwise, we say that $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ is ____________________________________.

Give some examples of a linearly independent set and a linearly dependent set.

**Example 8:** Is the set

$$\{(1, -2, 5, -3); \quad (2, 3, 1, -4); \quad (3, 8, -3, -5)\}$$

linearly independent in $\mathbb{R}^4$?