Math 20480 – Example Set 12A

1. Find the exponential of the matrix
$$A = \begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix}$$
by the following steps.

1a. Find the **complex** canonical form J associated to A including the transition matrix Q.

1b. Find the kth power of A in terms of Q and J.

1c. Let $T_k(x)$ be the kth Maclaurin's polynomial of e^x:

$$T_k(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}$$

Find a formula for $T_k(A)$ in terms of Q and J.

1d. By taking limit of your answer in (c), find e^A in terms of Q and J. Extract a real form for the exponential e^A.

2. Discuss the solution of the system of differential equations:

$$\begin{aligned} x'(t) &= -3\,x(t) - 4\,y(t); \qquad x(0) = 1 \\ y'(t) &= 2\,x(t) + y(t); \qquad y(0) = 2 \end{aligned}$$

3. Solve for the general solution of the system of differential equation:

$$\begin{aligned} x' &= -7x - 10y + 10z \\ y' &= 2x + y - 2z \\ z' &= -2x - 4y + 3z \end{aligned}$$

Math 20480 – Example Set 12B

Non-homogeneous Linear System of Differential Equations with Constant Coefficients. A system of differential equations takes the form

$$\vec{X}'(t) = A \cdot \vec{X}(t) + \vec{B}(t)$$

where A is a $n \times n$ real-valued matrix and $\vec{B}(t)$ is a $n \times 1$ vector-valued function. In two dimensions, we explicitly have

$$\begin{aligned} x'(t) &= \quad a\,x(t) + b\,y(t) + f(t) \\ y'(t) &= \quad c\,x(t) + d\,y(t) + g(t) \end{aligned}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{B}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$.

Such equations are solved using an integrating (matrix) factor $e^{-A \cdot t}$. Multiply the equation by $e^{-A \cdot t}$ and discuss how the system may be solved.

1. Recall Richardson's Theory of Conflict for two people groups G_1 and G_2. Let $x(t)$ be the potential for war (armament) for G_1 and $y(t)$ be the potential for war for G_2. Then we have:

$$\frac{dx}{dt} = \quad k\,y - \alpha\,x + g$$

$$\frac{dy}{dt} = \quad l\,x - \beta\,y + h$$

Richardson estimated that for the two allies G_1 : Germany-Austria-Hungary and G_2 : France-Russia that $k = 0.9 = l$ and $\alpha = 0.2 = \beta$. Here α^{-1} is the lifetime of the parliament so in this case we are taking the lifetime of the parliament (governing body) to be 5 years for both groups. Also k is proportional to the amount of industry a nation has. Germany grew its armament to level of its neighboring countries over the period of 1933-1936 so $k \approx 0.3$. Since both allies are about the same size (about three times that of Germany), we take $k = 0.9 = l$. If $g = 1.4 = h$, the system is:

$$\frac{dx}{dt} = \quad -0.2\,x + 0.9\,y + 1.4$$

$$\frac{dy}{dt} = \quad 0.9\,x - 0.2\,y + 1.4$$

Find x and y and interpret your results.

Matrix Exponential Formulas (Real Eigenvalue Case).

$$exp\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} = \begin{pmatrix}e^{\lambda} & 0\\ 0 & e^{\mu}\end{pmatrix}; \qquad exp\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} = \begin{pmatrix}e^{\lambda} & e^{\lambda}\\ 0 & e^{\lambda}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & 0\\ 0 & \mu t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{\mu t}\end{pmatrix};$$

$$exp\left[\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t\\ 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}$$

$$exp\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} = \begin{pmatrix}e^{\lambda_1} & 0 & 0\\ 0 & e^{\lambda_2} & 0\\ 0 & 0 & e^{\lambda_3}\end{pmatrix}; \qquad exp\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix} = \begin{pmatrix}e^{\lambda} & e^{\lambda} & e^{\lambda}\\ 0 & e^{\lambda} & e^{\lambda}\\ 0 & 0 & e^{\lambda}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\right]\cdot t = exp\begin{pmatrix}\lambda_1 t & 0 & 0\\ 0 & \lambda_2 t & 0\\ 0 & 0 & \lambda_3 t\end{pmatrix} = \begin{pmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & e^{\lambda_2 t} & 0\\ 0 & 0 & e^{\lambda_3 t}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t & 0\\ 0 & \lambda t & t\\ 0 & 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}\\ 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & e^{\lambda t}\end{pmatrix}$$

$$exp\left[\begin{pmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}\lambda t & t & 0\\ 0 & \lambda t & t\\ 0 & 0 & \lambda t\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}\\ 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & e^{\lambda t}\end{pmatrix}$$

Matrix Exponential Formulas (Complex Eigenvalue Case).

$$exp\begin{pmatrix}a & b\\ -b & a\end{pmatrix} = \begin{pmatrix}e^a\cos(b) & e^a\sin(b)\\ -e^a\sin(b) & e^a\cos(b)\end{pmatrix}$$

$$exp\left[\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\cdot t\right] = exp\begin{pmatrix}at & bt\\ -bt & at\end{pmatrix} = \begin{pmatrix}e^{at}\cos(bt) & e^{at}\sin(bt)\\ -e^{at}\sin(bt) & e^{at}\cos(bt)\end{pmatrix}$$

Remark 1. Let J be the canonical form of square matrix A and non-singular P such that $A = PJP^{-1}$. Then $e^A = P\cdot e^J\cdot P^{-1}$.

Remark 2. If $D = diag(A_1, A_2, \ldots, A_k)$ where A_i is a square matrix of size n_i so D is a square matrix of size $N = \sum n_i$. Then we have:

$$e^D = diag(e^{A_1}, e^{A_2}, \ldots, e^{A_k}).$$