1. More examples

Example 1. *Find the slope of the tangent line at* $(3,9)$ *to the curve* $y = x^2$.

$P = (3,9)$. So $Q = (3 + \Delta x, 9 + \Delta y)$.

$$9 + \Delta y = (3 + \Delta x)^2 = (9 + 6\Delta x + (\Delta x)^2.$$

Thus, we have

$$\frac{\Delta y}{\Delta x} = 6 + \Delta x.$$

If we let Δ go to zero, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 6 + 0 = 6.$$

Thus, the slope of the tangent line at $(3,9)$ is 6.

Let's consider a more abstract example.

Example 2. *Find the slope of the tangent line at an arbitrary point* $P = (x,y)$ *on the curve* $y = x^2$.

Again, $P = (x,y)$, and $Q = (x + \Delta x, y + \Delta y)$, where $y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$. So we get,

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Letting Δx go to zero, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x + 0 = 2x.$$

Thus, the slope of the tangent line at an arbitrary point (x,y) is $2x$.

Example 3. *Find the slope of the tangent line at an arbitrary point* $P = (x,y)$ *on the curve* $y = ax^3$, *where* a *is a real number.*

$Q = (x + \Delta x, y + \Delta y)$. We get,

$$y + \Delta y = a(x + \Delta x)^3 = a(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3).$$

After simplifying we have,

$$\frac{\Delta y}{\Delta x} = 3ax^2 + 3xa\Delta x + a(\Delta x)^2.$$

Letting Δx go to 0 we conclude,

$$m_P = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = 3ax^2 + 0 + 0 = 3ax^2.$$

Thus, the slope of the tangent line at a point (x, y) is $m_P = 3ax^2$.

Example 4. *Find the slope of the tangent line at an arbitrary point $P = (x, y)$ on the curve $y = mx + b$.*

The curve is a line. We expect the tangent line to have slope m. $y + \Delta y = m(x\Delta x) + b = mx + m\Delta x + b$. After simplifying, we get

$$\frac{\Delta y}{\Delta x} = m.$$

Thus, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = m$, which confirms our expectations.

Example 5. *Find the slope of the tangent line at an arbitrary point $P = (x, y)$ on the curve $x = y^2$.*

Here we have, $x + \Delta x = (y + \Delta y)^2 = y^2 + 2y\Delta y + (\Delta y)^2$. After simplifying, we get

$$\Delta x = 2y\Delta y + (\Delta y)^2 = \Delta y(2y + \Delta y).$$

This give us,

$$\frac{\Delta y}{\Delta x} = \frac{1}{2y + \Delta y}.$$

As Δx goes to 0, notice that Δy goes to 0 as well. Thus, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2y}.$$

Thus, the slope of the tangent line at (x, y) is $\frac{1}{2y}$ or $\frac{1}{2\sqrt{x}}$.

2. Derivatives

Let $f(x)$ be a function and let $P = (x, f(x))$ be a point on its graph. Let $Q = (x + \Delta x, f(x + \Delta x))$ be another point on the graph near P. As we did in the last couple sections, we will push Q closer to P. Recall that we defined the slope of the tangent line at P to be

$$m_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

We define the *derivative of* $f(x)$ to be the function $f'(x)$ given by the rule,

$$f'(x) = m_x.$$

Again, m_x is the slope of the tangent to the graph at a point $(x, f(x))$.

Once more, for good measure, for a function $f(x)$, the **derivative of** $f(x)$ is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Notice that if $y = f(x)$, we have that $f(x+\Delta x) - f(x) = y\Delta y$. So $f'(x)$ is precisely the slope of the tangent line at $(x, f(x))$. Let's consider a familiar example.

Example 6. *Let $f(x) = x^2$. Find $f'(2)$.*

By definition,

$$f'(x) = \lim_{\Delta x \to 0} \tfrac{f(2+\Delta x) - f(2)}{\Delta x}$$

So we have $\lim_{\Delta x \to 0} \frac{4+4\Delta x+(\Delta x)^2-4}{\Delta x} = \lim_{\Delta x \to 0} \frac{4\Delta x+(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2+\Delta x = 4$. Thus, the slope of the tangent line at $(2, 4)$ is 4 as we expected.

Example 7. *Let $f(x) = x^2$. Find $f'(x)$ for arbitrary x.*

By going through the same process as above, we'll get $f'(x) = 2x$.

3. Power rule

The definition of the derivative is a beautiful thing. However, it can be a little cumbersome if we have to go through the limiting process every time we would like to find the derivative of a function. Fortunately, there are many shortcuts, called differentiation rules, that make the process go by much more effectively. Of course, we can't just use them without exploring where they come from. The first such rule is the *power rule*, and in this section we will give the proof for positive rational powers.

Theorem 3.1 (Power rule for derivatives)**.** *Let $f(x) = x^r$ where r is a real number. Then$f'(x) = rx^{r-1}$.*

Proof. First let's go through the proof for when $f(x) = x^n$ for n a positive integer. Let's begin with the definition of the derivative. Remember Pascal's triangle and how it seemed completely unrelated to functions? Well, we're going to use it here.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \tfrac{(x+\Delta x)^n - x^n}{\Delta x}$$

We need to expand $(x+\Delta x)^n$. Fortunately, we have Pascal's triangle and Newton's binomial theorem. So we know that

$$(x+\Delta x)^n = \tbinom{n}{0}x^n + \tbinom{n}{1}x^{n-1}\Delta x + \cdots + \tbinom{n}{n}(\Delta x)^n.$$

$$= x^n + nx^{n-1}\Delta x + (\Delta x)^2(\text{ stuff }).$$

Above, we pulled $(\Delta x)^2$ out of all but the first two terms. Notice that we can do this. We write "stuff" because these terms will disappear very soon. Continuing on from above, we have the following :

$$\begin{aligned}
& \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
= \quad & \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + (\Delta x)^2(\text{ stuff }) - x^n}{\Delta} \\
= \quad & \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + (\Delta x)^2(\text{ stuff })}{\Delta x} \qquad . \\
= \quad & \lim_{\Delta x \to 0} nx^{n-1} + (\Delta x)(\text{ stuff }) \\
= \quad & nx^{n-1}
\end{aligned}$$

So we have proved the power rule for n an integer. Let's consider the proof for positive rationals. That is, $f(x) = x^{p/q}$. Let $y = x^{p/q}$. Then we have $y^q = x^p$. Thus, $(y+\Delta y)^q = (x+\Delta x)^p$. Now consider the following :

$$\begin{aligned}
(y+\Delta y)^q &= (x+\Delta x)^p \\
y^q + qy^{q-1}\Delta y + (\Delta y)^2(\text{stuff}) &= x^p + px^{p-1}\Delta x + (\Delta x)^2(\text{stuff}) \\
qy^{q-1}\Delta y + (\Delta y)^2(\text{stuff}) &= px^{p-1}\Delta x + (\Delta x)^2(\text{stuff}) \\
\Delta y(qy^{q-1} + (\Delta y)(\text{stuff})) &= \Delta x(px^{p-1} + (\Delta x)(\text{stuff})) \\
\frac{\Delta y}{\Delta x} &= \frac{px^{p-1} + (\Delta x)(\text{stuff})}{qy^{q-1} + (\Delta y)(\text{stuff})} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \frac{px^{p-1}}{qy^{q-1}} \\
&= \frac{px^{p-1}}{q(x^{\frac{p}{q}})^{(q-1)}} \\
&= \frac{px^{p-1}}{qx^{p-\frac{p}{q}}} \\
&= \frac{p}{q}x^{p-1-(p-\frac{p}{q})} \\
&= \frac{p}{q}x^{\frac{p}{q}-1}.
\end{aligned}$$

We should make a few quick notes about the process above. First, when we expanded $(y+\Delta y)^q$ and $(x+\Delta x)^p$, we again used Newton's binomial theorem as in the earlier case. After we took the limit as Δx went to 0, we replaced y with $x^{p/q}$. From there, we simply used algebra to get what we needed.

We should also prove the power rule for $f(x) = x^{-n}$ for a positive integer n. The proof is a slightly harder version of x^n. We will leave it as an exercise for the reader. To prove the power rule for irrational numbers, we just use the continuity for power functions. This sentence most likely does not make sense to most beginning calculus students. Do not worry about it; we include it only for the sake of completion. You'll see it again if you ever take a real analysis course. □

Example 8. *Let $f(x) = x^{35/4}$. Find $f'(x)$.*

From the power rule, we have that $f'(x) = \frac{35}{4}x^{35/4-1} = \frac{35}{4}x^{31/4}$.

We should notice something nice about the power rule. If $f(x) = x^r$, to find $f'(x)$, all we need to do is move the exponent down and subtract 1 from the exponent. That is $f'(x) = rx^{r-1}$.