1. Local extrema

We finally have all the tools we need for finding derivatives. We will set the computation aside for just a bit to see how derivatives can help us analyze a function's characteristics. For example, consider the function $f(x) = x^2$. Here are two simple questions about $f(x)$. Does $f(x)$ have a smallest value? (i.e., is there a c such that $f(c) \leq f(x)$ for any other x?) Does it have a largest value? (i.e., is there a c such that $f(c) \geq f(x)$ for all x?)

Consider the graph of the function. It is shaped roughly like a large U. We can see that there is a smallest value, and it occurs at $x = 0$. The actual smallest value is then of course $f(0) = 0$. What about a maximum value? Again, it is not hard to believe that there is no maximum value. The "U" rises up to infinity. It is very normal for functions not to have smallest and largest values. We call such points *global extrema.*

We are interested in this smallest value at $x = 0$. Let's consider the shape of the graph near that point. Let's ask, what is the slope of the tangent at the point $x = 0$? We can see from the graph that it is very horizontal. We can estimate it to be 0, but let's make sure with derivatives. $f'(x) = 2x$. So at $x = 0$, $f'(0) = 0$. The slope is indeed 0, so it is horizontal.

So at this minimum value, we see that the derivative is 0. What else can we determine? If we think an interval around 0, we can see that the minimum is at the bottom of a "valley." This makes since, right? For a function to have a minimum value, it will have to "dip down" to create it. So it is at the bottom of a valley. What else?

Consider the slopes of the tangents to the left and to the right of 0. Either by looking at the graph or taking the derivative, we see that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. What does this mean? It means that the function is *deceasing* on the left side of the minimum value and is *increasing* on the right side of the minimum value. So the derivative *changes sign* from negative to positive at the point $x = 0$.

Let's consider $f(x) = \sin(x)$. Does this function have global extrema? Sure it does. It even has infinitely many. It has a maximum value at $x = \pi/2, -3\pi/2, 5\pi/2, ...$ It has a minimum value at $3\pi/2, -\pi/2, 7\pi/2, ...$ Of course, the smallest value is -1 and the largest value is 1. Let's just consider the interval $0 \leq x \leq \pi$.

Since we are no longer considering the whole function, we discuss the "local" properties of the function. "Local" is a fancy word meaning we are concerning ourselves with just a piece of the whole function. Any largest and smallest values in this restricted interval are called *local extrema*. Locally, we have a local maximum at $x = \pi/2$. Do we have any minimums?

It turns out that we do. *Locally* the function has smallest values at $x = 0$ and $x = \pi$. Remember, this means only on the interval $[0, \pi]$, there are *local* minimums at 0 and π.

Let's look at our three special points more closely. At $x = \pi/2$, we have a local maximum that looks a lot like the minimum of x^2. We see that $f'(\pi/2) = \cos(\pi/2) = 0$, so it also has a tangent of 0. It occurs at the "top of a hill." On the left, the derivative is positive, and to the left it is negative. So the function increases to the maximum and then decreases.

The other two points are different. Notice that they are not bottoms of valleys. This is because they are the *endpoints* of our interval. Globally, they are not minimums, but locally they are. Hopefully, without too much convincing, we can believe the following.

Proposition 1.1. *Let $f(x)$ be a differentiable function on an interval $[a, b]$. Suppose that $f(c)$ is a local maximum (or minimum). Then either $c = a$ or $c = b$, or $f'(c) = 0$.*

This proposition says that local extrema can only occur where a function has a horizontal tangent line or at endpoints of intervals. Notice that if our interval is $(-\infty, \infty)$, then there are no endpoints.

We should also note that the converse is not true. It is not the case that if c is an endpoint or $f'(c) = 0$ then $f(c)$ is a local minimum or maximum. For example, consider $f(x) = x^3$. We have that $f'(0) = 0$, but looking at the graph, we see that at $x = 0$, there is no maximum or minimum. So even though the slope of the tangent line is 0 at $x = 0$, we do not have an extrema there. For another example, consider $f(x) = \sin(x)$ on $[0, \pi]$. The values at both endpoints are 0, but the local minimum on this interval is $f(3\pi/2) = -1$. So endpoints are not necessarily extrema.

The general method for find local extrema in a graph $f(x)$ on an interval $[a, b]$ is to take the derivative $f'(x)$. We then solve $f'(x) = 0$. Solutions to that equation along with a and b are what we call "critical points." These points are candidates for being extrema. We can check by either plugging in values or determining how the function behaves near these points (increasing, decreasing, etc.).

2. Mean value theorem

Consider a function $f(x)$ on an interval $[a, b]$. Let $m = \dfrac{f(b) - f(a)}{b - a}$ be the slope of the line connecting the two endpoints. The question is, what can we say about the values for $f'(x)$ on this interval?

Theorem 2.1 (Mean value theorem)**.** *Let $f(x)$ be a differentiable function on the interval $[a, b]$. Let $m = \dfrac{f(b) - f(a)}{b - a}$. Then there is some point c such that $a < c < b$ and $f'(c) = m$.*

We can think of m as the *average rate of change* from $f(a)$ to $f(b)$. The mean value theorem says that there is some point between a and b where the slope of the tangent line at that point is exactly the average rate of change.

We will not prove the mean value theorem. It is easy to convice yourself that it's true though. Consider a nice, smooth graph of a function on some interval. Imagine drawing a curve between the endpoints. It makes sense that at least once, the curve will have a tangent with slope the average rate of change.