1. Properties of ellipses

Recall that given two foci F_1 and F_2, and a number $k \geq F_1F_2$, the set of points P such that $PF_1 + PF_2 = k$, is a conic section called an *ellipse*. Last time we saw that for an ellipse centered at the origen with foci $(-e, 0)$ and $(e, 0)$, its corresponding equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = \frac{k}{2}$ and $b = \sqrt{a^2 - e^2}$. We will look at some of the basic properties of the ellipse with reference to the cartesian plane.

By setting $y = 0$, we see that $x^2 = a^2$, or $x = \pm a$. So the $x-$intercepts for an ellipse are $(-a, 0)$ and $(a, 0)$. Again, if $e = a$, we have the the foci are the endpoints for the line segment created. Therefore, we see that the length of the *major axis* is $2a$.

By setting $x = 0$, we get $y^2 = b^2$, or $y = \pm b$. So the y-intercepts for an ellipse are $(0, -b)$ and $(0, b)$. This means that the length of the *minor axis* is $2b$.

Suppose that $e = 0$. This means that the foci are both at the origen $(0, 0)$. This also means that $b = \sqrt{a^2 - 0} = a$. So the lengths of the major and minor axes are the same, meaning that our ellipse is a circle of radius $a = k/2$.

The number e, which is the distance between the origen and either focus, is called the *linear eccentricity* of the ellipse. Refer to figure 4.28 in your text. Since $b^2 = a^2 - e^2$ or $a^2 = b^2 + e^2$, by the Pythagorean theorem, we have that the length from a focal point to either $(0, -b)$ or $(0, b)$ is a. We call the ratio

$$\varepsilon = \frac{\text{linear eccentricity}}{\text{semimajor axis}} = \frac{e}{a}$$

is called the *astronomical eccentricity*. Since $a \geq e$, it follows that $0 \leq \varepsilon \leq 1$.

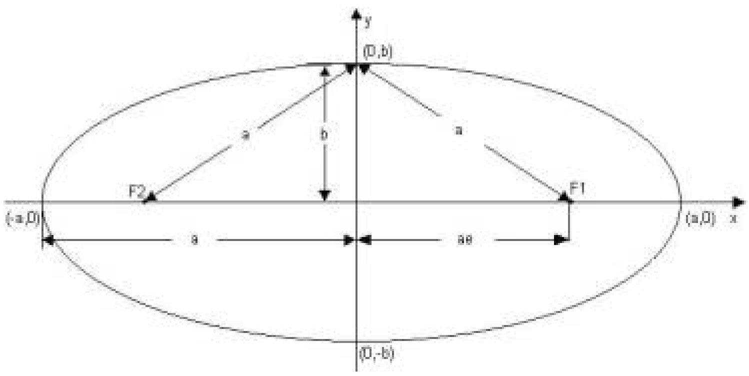

Figure 1

We can think of ε as a measurement of the "flatness" of an ellipse. Suppose, $e = 0$. We know that this gives us a circle. Also notice that $\varepsilon = 0$, which means the circle has "no flatness." Now suppose that $a = e$. We know that this is just the line segment between $(-e, 0)$ and $(e, 0)$. Also notice that $\varepsilon = 1$, which means the line as "maximum flatness." Most ellipses have astronomical eccentricity between 0 and 1, which will yield an oval shape. What we can take from this is that if an ellipse is close to being a circle, then b is close to a. If the ellipse is very flat, then b is relatively small compared to a.

Example 1. *Show that the graph of* $9x^2 + 16y^2 = 144$ *is an ellipse.*

Divide both sides of the equation by 144 to get,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is an equation for an ellipse with $a = 4$ and $b = 3$. Thus, $e = \sqrt{a^2 - b^2} = \sqrt{7}$. So it is an ellipse with foci $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$, x-intercepts $(-4, 0)$ and $(4, 0)$, and y-intercepts $(0, -3)$ and $(0, 3)$. Furthermore, the astronomical eccentricity is $\frac{\sqrt{7}}{4} \approx 0.66$.

2. Cavalieri's principle

Suppose you have two regions on the plane with areas C and D. Place them above the x-axis as picture in figure 3 below or figure 4.29 in your text. Further, suppose that for every x, the vertical cross sectional that cuts through C is the same as the one that cuts through D. That is, $c_x = d_x$ for all x. Then we have that $C = D$. The intuition behind this can be obtained by thinking of splitting the regions into very small strips. If for every vertical strip in C, the corresponding strip in D has the same area, then the total areas C and D will be the same. Figure 2 gives another view of this idea.

Instead, suppose that for each x, $d_x = 2c_x$. Then it makes sense to conclude that $D = 2C$. If $d_x = 1{,}000c_x$, then it should be the case that $D = 1{,}000C$. It should not be difficult to convince yourself of the following proposition :

Proposition 2.1 (Cavalieri's Principle)**.** *For a positive number k, if $d_x = kc_x$ for all x, then $D = kC$.*

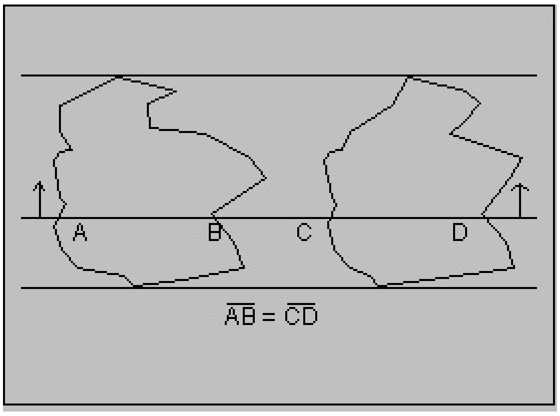

Figure 2

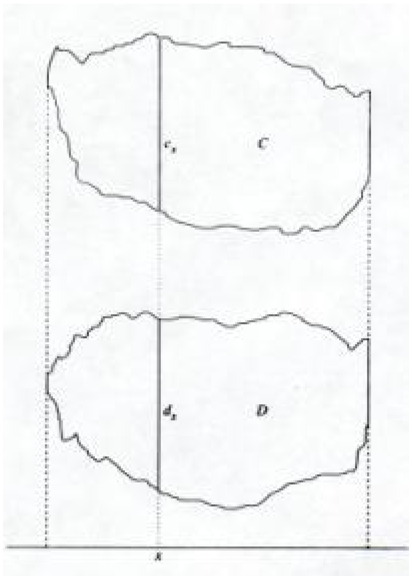

Figure 3