Math 105 **Answers to test 3**

Part I. Multiple Choice

1. $\frac{d}{dx}(x^3e^{4x}) = 3x^2e^{4x} + 4x^3e^{4x}$ by the product rule.

2. Suppose $f(x)$ is a function which satisfies $f'(3) = 0$, $f'(5) = 0$, $f''(3) = -4$, and $f''(5) = 5$. Which of the following statements is true? **Ans.** $f(x)$ has a relative maximum at $x = 3$ and a relative minimum at $x = 5$. (The second derivative test)

3. Let $y = \sqrt{\ln x}$. Then $y' = \frac{1}{2x\sqrt{\ln x}}$ by the chain rule.

4. Let $h(x) = \frac{f(x)}{g(x)}$. Then $h'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = -\frac{5}{8}$. (Use the given table.)

5. Let $k(x) = f(g(x))$. Then $k'(0) = f'(g(0)) \cdot g'(0) = f'(2) \cdot g'(0) = 1$. (Use the given table.)

6. Let $f(x)$ be a function with $f'(2) = 0$. Which of the following **must** be true about f? **Ans.** The tangent line to the graph of f at $(2, f(2))$ is horizontal.

7. Suppose that $f(x)$ is increasing and concave up everywhere. Further, $f(2) = 2$ and the tangent line at the point $(2, 2)$ is the line $y = x$. Which of the following can you conclude? **Ans.** $f(5) > 5$. If the graph is concave up its slope is increasing, so it is above the tangent line $y = x$ for all $x > 2$.

8. The graph of a function $f(x)$ is sketched below. Which of the following statements is true? **Ans.** $f(x)$ is increasing and $f'(x)$ is decreasing.

9. Suppose that $f(x)$ is a function with $f''(x) > 0$ on the interval $(2, \infty)$ and $f''(x) < 0$ on the interval $(-\infty, 2)$. Which of the following could be the graph of $f(x)$? (Use the letter to the upper left of each graph.) Note that the answer is the only graph that is concave up for $x > 2$ and down for $x < 2$.

Ans.

10. Suppose that $f(x)$ is a function whose **derivative** is $f'(x) = x \ln x$. Identify the x-value(s) of all inflection point(s) of $f(x)$. **Ans.** $1/e$, because $f''(x) = \ln x + 1$. The only solution to $f''(x) = 0$ is $x = 1/e$, and one checks that $f''(x)$ changes sign there.

Part I. Partial Credit

11. A pot of boiling water (100° Celsius) is left to cool in a room whose temperature is 20° Celsius so that after t minutes, the temperature of the water is $H = 20 + Ae^{kt}$ for some constants A and k. After 30 minutes, the water is 60°. At what **rate** (with respect to time) is the temperature of the water changing after 90 minutes?
Ans. $-(\ln 2)/3 \approx -0.231$. First, substitute $t = 0$, and $H = 100$ to get $A = 80$. Then, substitute $t = 30$ and $H = 60$ and solve for k to get $k = -(\ln 2)/30$. The rate at which the temperature changes is $H'(t) = kAe^{kt}$. Using the values of A and k just determined and setting $t = 90$ gives the answer.

12. The half-life of carbon-14 is 5730 years, so carbon decays according to the equation $y = Ae^{-kt}$, where $k = .000121$. An ancient document written on papyrus is found to have 37.5 % of the carbon-14 that a freshly cut sample has. How old is the document?
Ans. $-(\ln(0.375))/0.000121 \approx 8106$ years. Because $0.375 = \dfrac{y(t)}{y(0)} = \dfrac{Ae^{-kt}}{A} = e^{-0.000121t}$. Solving for t gives the answer.

13. Find the the equation of the line tangent to $y = \dfrac{e^{x^2+2x}}{x^3+1}$ at $(0, 1)$. **Ans.** $y = 2x + 1$. First find y' by using the quotient and chain rules. Plugging in $x = 0$ gives $y'(0) = 2$, which is the slope of the tangent line.

14. The period P in seconds of a simple pendulum is completely determined by its length L according to the formula $P = \dfrac{2\pi}{\sqrt{10}}\sqrt{L}$, where $\pi \approx 3.14$. Therefore, if the length of a pendulum increases, its period will increase. (That is why clocks with metal pendulums tend to run slow in the summertime.) Find the rate (with respect to time) at which a simple pendulum's period is increasing when the pendulum is 4m long and its length is increasing at a rate of .001m/day. **Ans.** $\dfrac{0.001\pi}{2\sqrt{10}} \approx 0.0004967$. Use the chain rule to get

$$\frac{dP}{dt} = \frac{2\pi}{\sqrt{10}} \cdot \frac{1}{2\sqrt{L}} \cdot \frac{dL}{dt} = \frac{\pi}{\sqrt{10}} \cdot \frac{1}{\sqrt{L}} \cdot \frac{dL}{dt}.$$

Then plug in $L = 4$ and $dL/dt = 0.001$ to get the answer.

15. The demand q for Widgets is a function of the price p, i.e. $q = f(p)$. When the price $p = \$10$, we have $q = f(10) = 100$ and $f'(10) = -4$.

(a) The revenue R is determined by $R = pq$. Find dR/dp when $p = 10$. **Ans.** 60. Write $R = pf(p)$, so by the product rule $R'(p) = pf'(p) + f(p)$. Then plug in $p = 10$.

(b) The cost of producing q Widgets is $C(q) = 100q + 5000$. Find dC/dp when $p = 10$. **Ans.** -400. By the chain rule, $dC/dp = (dC/dq) \cdot (dq/dp) = 100f'(p)$. Then plug in $p = 10$.

(c) If P represents profit, find dP/dp when $p = 10$. **Ans.** 460. Use $P = R - C$, so that $dP/dp = dR/dp - dC/dp$.

16. Consider the function $f(x) = x - 3x^{\frac{2}{3}}$.

(a) Identify the intervals on which $f(x)$ is increasing and those on which $f(x)$ is decreasing.

(b) Identify the x-value(s) of all relative extreme point(s) of $f(x)$ and specify whether each is a maximum or a minimum.

Ans. (a) $f(x)$ is increasing on $(-\infty, 0)$ and on $(8, \infty)$, decreasing on $(0, 8)$. (b) There is a relative max at $x = 0$ and a relative min at $x = 8$. First compute the derivative, $f'(x) = 1 - 2x^{-\frac{1}{3}}$. There are two critical points: at $x = 0$, where $f'(x)$ does not exist, and at $x = 8$, where $f'(x) = 0$. Checking the sign of $f'(x)$ on the intervals $(-\infty, 0)$, $(0, 8)$, and $(8, \infty)$ gives the answer.

17. Consider the function $f(x) = x^4 - 54x^2 + 27x - 17$.

(a) Identify the intervals on which $f(x)$ is concave up and those on which $f(x)$ is concave down.

(b) Identify the x-value(s) of all inflection point(s) of $f(x)$.

Ans. (a) Concave up on $(-\infty, -3)$ and on $(3, \infty)$, concave down on $(-3, 3)$. (b) Inflection points at $x = 3$ and $x = -3$. Compute the second derivative, $f''(x) = 12x^2 - 108$, and solve $f''(x) = 0$ to get $x = \pm 3$. Checking the sign of $f''(x)$ on the intervals $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$ gives the answer.

18. Suppose that $f(x)$ is a function which satisfies the following list of properties:

- $f(-3) = -1$, $f(0) = -3$, $f(2) = -5$, and $f(4) = 0$.
- The only solutions to $f'(x) = 0$ are $x = -3$ and $x = 2$.
    - $f'(x) > 0$ on the intervals $(-\infty, -3)$, $(2, \infty)$.
    - $f'(x) < 0$ on the interval $(-3, 2)$.
- The only solution to $f''(x) = 0$ is $x = 0$.
    - $f''(x) > 0$ on $(0, \infty)$.
    - $f''(x) < 0$ on $(-\infty, 0)$.

(a) List all the relative extreme points and inflection points of $f(x)$.

(b) Sketch a possible graph of $f(x)$, labeling the points you listed in part (a).

Ans. Relative maximum at $x = -3$, relative minimum at $x = 2$, inflection point at $x = 0$. The graph is drawn below.