Math 105, Spring '99 **Supplementary Problems**

1. A tour boat operator found that when the price charged for the scenic boat tour was \$25, the average number of customers per week was 500. When the price was reduced to \$20, the average number of customers per week went up to 650. Assuming that the number of customers is a linear function of the price, find the equation and graph it. (A graph like this is called a *demand curve*, because it represents the demand as a function of the price.)

2. A private health club has determined that its cost and revenue functions are given by

$$C = 10,000 + 35q \quad \text{and} \qquad R = pq,$$

where q is the number of annual club members and p represents the price of a one-year membership. The numbers p and q are related by the demand equation

$$q = 3000 - 20p. \tag{1}$$

(a) Solve for p in terms of q.

(b) Write the revenue and profit as functions of q, and find the break-even points in terms of q.

(c) Economists often take the price as the independent variable, rather than the number sold, since the price can be more easily controlled by the company. Using equation **??**, write the revenue, cost and profit as functions of p. For what range of prices does the club make a profit?

3. A sporting goods wholesaler finds that when the price of one of the products is \$25, the company sells an average of 500 units per week. When the price is \$30, the average number sold per week decreases to 460 units.

(a) If we assume that the demand function for this product is linear, find the demand q as a function of the price p.

(b) Use your answer to part (a) to write the revenue R as a function of the price p.

4. Complete the square of each of the following quadratic equations. Then state whether the graph opens upward or downward, and find the low or high point. (Find both coordinates of the point.)

(a) $y = -x^2 + 6x + 11$ (b) $y = 3x^2 + 6x - 1$

(c) $y = 2x^2 + 5x$ (d) $y = -2x^2 + x + 1$

5. A baseball is thrown from ground level straight up into the air at time $t = 0$ with an initial velocity of 64 feet per second. Its height h at time t is given by the formula

$$h = -16t^2 + 64t\,.$$

(a) At what time does the ball hit the ground?

(b) At what time does it reach its maximum height? In other words, for what t does the graph have its high point?

(c) What is the maximum height it reaches?

6. In problem 2, what price should the club charge to maximize: (a) its revenue, and (b) its profit?

7. In problem 3, what price should the wholesaler charge to maximize its revenue?

8. What are the horizontal and vertical asymptotes (if any) of the graphs of the following functions? If there is a vertical asymptote, determine how the graph behaves (i.e., climbs or falls) as it approaches the asymptote from either side.

(a) $f(x) = 1/(x+1)^2$ (b) $f(x) = 1/(x-1)^3$

(c) $f(x) = (2x-1)/(x+4)$ (d) $f(x) = x/(x^2-4)$

(e) $f(x) = (x^2-3)/(3x^2+1)$ (f) $f(x) = x^2/(x-2)$

9. According to the April 1991 issue of *Car and Driver*, an Alfa Romeo going at 70 mph requires 177 feet to stop. Assuming that the stopping distance is directly proportional to the square of the velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph (its top speed).

10. (See example 6, p. 7 of the textbook). A leading brokerage firm charges a 6% commission on gold purchases in amounts from \$50 to \$300. (It does not deal with amounts less than \$50.) For purchases exceeding \$300 up to and including \$600, the firm charges 2% of the amount purchased plus \$12.00. For purchases exceeding \$600, it charges 1.5% of the amount purchased plus \$15. Express the brokerage fee as a function of the amount of gold purchased. Is the function continuous?

11. State whether or not each of the following functions is continuous. If not, where are its points of discontinuity?

(a) $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2-x & \text{if } 0 \le x < 1 \\ x & \text{if } x \ge 1 \end{cases}$ (b) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } 0 \le x < 2 \\ x/2 & \text{if } x \ge 2 \end{cases}$

12. Solve the following equations for t using natural logarithms.

(a) $5^t = 7$ (b) $10 = 2^t$ (c) $e^{3t} = 100$ (d) $5e^{3t} = 8e^{2t}$

13. Simplify each of the following expressions as much as possible.

(a) $\ln(1/e^3)$ (b) $e^{\ln(x+1)}$ (c) $e^{2\ln x}$ (d) $3\ln e + \ln(1/e)$

14. Rewrite each of the following as an exponential with base e.

(a) 3^7 (b) 5^x (c) x^5

15. Rewrite each of the following as a quotient of natural logarithms. Then use your calculator to find a decimal approximation.

(a) $\log_5 12$ (b) $\log_{12} 5$ (c) $\log_2 e$

16. It is believed by some that the earth's population cannot exceed 40 billion people and can be modeled by the function $P = 40/(1 + 11e^{-0.08t})$. In this formula, P is the earth's population in billions and t is the number of years after 1990 (i.e., 1991 corresponds to $t = 1$, 1992 corresponds to $t = 2$, etc.). Assuming this model is valid, answer the following questions.

(a) What will the earth's population be in the year 2000?

(b) When will the earth's population reach 20 billion?

17. If \$12,000 is deposited in an account paying 8% interest per year, compounded continuously, how long will it take for the balance to reach \$20,000?

18. Suppose you invest \$5,000 in an account that pays interest compounded continuously.

(a) How much money is in the account after 8 years if the annual interest rate is 4%?

(b) If you want the account to contain \$8,000 after 8 years, what annual interest rate is needed?

19. Use the secant approximation, that is compute

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

to find the slope of the graph $y = f(x)$ at the point $(a, f(a))$ in each of the following cases.

(a) $f(x) = x^2 + 3x,\ a = 1$ (b) $f(x) = x^2 + 3x,$ any a

(c) $f(x) = 1/(3x+1)\,,\ a = 0$ (d) $f(x) = \sqrt{2x+1},\ a = 4$

20. Use the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and no other derivative formulas to find $f'(x)$ for each of the following functions.

(a) $f(x) = x^2 + x - 1$ (b) $f(x) = 1/(2x+3)$ (c) $f(x) = 1/x^2$

21. Use the power rule to find the derivative of each of the following functions.

(a) $f(x) = x^{5/3}$ (b) $f(x) = 1/x^3$ (c) $f(x) = 1/\sqrt{x}$

22. A ball is tossed into the air from a window, and its height y (in feet) above the ground t seconds after it is thrown is given by

$$y = f(t) = -16t^2 + 50t + 36.$$

(a) At what height did the ball start?

(b) With what velocity did it start?

(c) What is its average velocity for the first second?

(d) What is its instantaneous velocity at $t = 1$ second?

(e) What is its instantaneous velocity at $t = 2$ seconds?

(f) The answers to (d) and (e) have different signs. Explain the significance of the signs.

(g) When does the ball have zero velocity? What does that mean in terms of where it is?

23. Find the derivatives of the following functions.

(a) $y = 2t^3 + 3\ln t$ (b) $f(x) = x^2 + 4x + \ln x$ (c) $y = 10 - \ln x$

24. Use the dervative formula for $\ln x$ and the laws of logarithms to find the derivatives of the following functions.

(a) $f(x) = \ln(x/5)$ (b) $f(x) = \ln(5/x)$ (c) $f(x) = \dfrac{\ln x}{5}$

(d) $y = \ln\sqrt{x}$ (e) $f(t) = \ln(2t^3)$ (f) $\ln(2/t^3)$

25. Find each of the following points (both coordinates).

(a) The point where the graph of $y = \ln x$ has slope 2.

(b) The point where the tangent to the graph of $y = \ln x$ is parallel to the line $x - 3y = 7$.

26. (a) Find the equation of the line tangent to the graph of $y = \ln x$ at $x = 1$.

(b) Use linear approximation to find approximate values for ln(1.1), ln(1.02), and ln(0.99), and compare them with the values you get from your calculator.

(c) If you did the approximations in part (b) correctly they were all greater than the values given by your calculator. Can you explain why? (Hint: sketch the graph of $y = \ln x$ and draw the tangent line at the the point $(0, 1)$. Compare the tangent line to the graph.)

27. Find the derivatives of the following functions.

(a) $y = 3e^t - t^2 + 5t + 1$ (b) $f(x) = 2e^x - \dfrac{1}{\sqrt{x}}$ (c) $y = e^x - \ln x + 1$

28. Find the following points (both coordinates).

(a) The point where the graph of $y = e^x$ has slope 3.

(b) The point where the tangent to the graph of $y = e^x$ is parallel to the line $x - 2y = 5$.

(c) The point where the line tangent to the graph of $y = e^x$ at the point $(0, 1)$ meets the x-axis.

29. Use the derivative formula for e^{ax} and the laws of exponents to find $f'(x)$ in each of the following cases.

(a) $f(x) = e^{x/2}$ (b) $y = (e^x)^4$ (c) $f(t) = \dfrac{1}{e^t}$

(d) $y = \sqrt{e^t}$ (e) $(e^x)^3 - 5\,(e^x)^2 + 3e^x - 1$ (f) $f(x) = (e^x - 3)^2$

30. Find the derivatives of the following functions.

(a) $f(t) = 3^t + t^3$ (b) $y = \dfrac{2^x}{3} + \dfrac{2}{3x}$ (c) $f(x) = 5^x + 5 \cdot 4^x$

31. Find the following points (both coordinates).

(a) The point where the graph of $y = 2^x$ has slope 1.

(b) The point where the tangent line to graph of $y = 2^x$ at the point $(0, 1)$ meets the x-axis.

32. With an inflation rate of 5%, prices are described by $P = P_0(1.05)^t$, where t is the time in years, P is the price in dollars, and P_0 is the price when $t = 0$. If this model is correct, how fast are prices rising (in dollars per year) when $t = 1$? When $t = 2$?

33. The population of Hungary has been decreasing in recent years at the rate of about 0.2% per year. Assuming this trend continues, the population can be approximated by the formula $P = 10.8(0.998)^t$, where t is the number of years since 1990 and P is the population in millions. According to this model, what will Hungary's population be in the year 2000, and how fast (in people per year) will it be decreasing?

34. The temperature of a refrigerator is maintained at 40 degrees Fahrenheit, and a can of soda is put into it to cool. The temperature of the can is 70 degrees when it is put into the refrigerator. According to Newton's law of cooling, the temperature H of the can t hours after it is put in the refrigerator is given by $H = 40 + 30e^{kt}$ for some constant k.

(a) Suppose the can cools down to 50 degrees in 1/2 hour. What is the constant k? Use a calculator to find the answer to 2 decimal places.

(b) Find the derivative dH/dt as a function of t. What is its sign? Why should you have expected that sign?

(c) For $t \geq 0$, when is the absolute value of dH/dt greatest? Why should you have expected that?

35. Suppose you put a yam in a hot oven maintained at a constant temperature of 200 degrees Celsius. Suppose that after 30 minutes the temperature of the yam is 120 degrees and is increasing at an (instantaneous) rate of 2 degrees per minute. Newton's law of cooling (or, in this case, heating) says that the temperature H of the yam at time t is given by a formula of the form $H = 200 + Ae^{Bt}$ for some constants A and B. If t is measured in minutes, find A and B. (Note: both A and B are negative.)

36. The quantity q of a certain skateboard sold depends on the selling price p, so we write $q = f(p)$. Suppose you know that $f(140) = 15,000$ and $f'(140) = -100$.

(a) What does this information tell you about the demand for skateboards when the price is \$140?

(b) The total revenue, R, earned by the sale of skateboards is given by $R = pq$. Find dR/dp when $p = 140$.

(c) What is the sign of dR/dp when $p = 140$? If the skateboards are currently selling for \$140, should the price be increased or decreased to increase revenue?

37. Assume the demand function for a certain product is given by

$$q = 1000e^{-0.02p}$$

where p is the price of the product and q is the quantity sold at that price.

(a) Write the revenue R as a function of the price p.

(b) Find the marginal revenue function.

(c) Find the revenue and marginal revenue when the price is \$10.

38. For each of the following functions, find the critical points and use the derivative to determine where the function is increasing, where it is decreasing, and where it has a local minimum or maximum.

(a) $f(x) = 2x^3 + 3x^2 - 36x + 5$ (b) $f(x) = 3x^4 - 4x^3 + 6$

(c) $f(x) = (x^2 - 4)^7$ (d) $f(x) = x - \ln x$ for $x > 0$

(e) $f(x) = xe^x$ (f) $= 2x^2e^{5x} + 1$

39. Let $f(x) = x^5 + x + 7$. Observe that $f(-2) < 0$ and $f(0) > 0$, so that $f(x) = 0$ for some x between -2 and 0. How many solutions does the equation $x^5 + x + 7 = 0$ have? Explain. (Hint: How many critical points does $f(x)$ have?)

40. Show that if x is any positive number, then $x + 1/x \geq 2$. (Hint: find the minimum of $f(x) = x + 1/x$.)

41. (a) Show that $x > 2\ln x$ for all $x > 0$. (Hint: find the minimum of $f(x) = x - 2\ln x$.)

(b) Use the previous result to show that $e^x > x^2$ for all $x > 0$.

(c) Is $x > 3\ln x$ for all $x > 0$?