Class Activities

Math 108, Spring 1999

Introduction. The activities below were used during the spring semester of 1999 and their main purpose was to encourage participation and active learning in class. The first thing I did when I walked in the first class was to distribute Activity 1, which introduces antidifferentiation, and asked the students to start working on it. Also, I told them that

- They should work in groups of 2-4
- Raise hand if their group needs help
- One paper with all names listed on it will be collected from each group and it will be graded.

While the students were working on the first activity, I finished distributing other handouts (general information about the course, and the first assignment), and walking around the room I provided guidance to groups that asked for help. Some groups finished fast, others took longer time than I expected. In any case, after about 15 minutes all groups finished and then I collected a paper from each group.

This first interaction with the students gave me valuable information about the class which I used immediately in my lecture about antidifferentiation. The rest of the class went the traditional lecturing style interrupted by a few questions from students and a few questions from the teacher. At the end of the class I informed the students that there will be activities similar to the one we had today in most classes of this course. In addition, it was stressed that

- They should read the material before is discussed in class with the objective to obtain an initial understanding of the concepts involved, and to recognize possible difficulties needed to be resolved in class.

Although many students at the beginning did follow the suggestion of reading before the material was considered, only a few continued to read till the end of the semester. It seems that pure encouragement alone does not work. Giving a simple quiz at the beginning of each class, testing reading and whose contribution in the course grade is significant, could work. However this was not an option since Math 108 is a multisection course having a common grading scheme, and the other seven sections were not using activities. The activities' grade was part of the homework grade. And the homework contributed only 50 points in the total of 500 points.

Taking into consideration that most students did not read the material to be discussed before class, the activities given fall into the following three kinds.

1. Activities based mostly on previous knowledge and on the new concepts, which could be easily described in the activity. Such examples are Activities 1, 2, 14.

2. Activities introducing the concept(s) to be considered in the lecture to follow in concrete situations. Such examples are Activities 1, 4, 18.

3. Activities applying the concepts and methods introduced in the lecture, and also demonstrated afterwards with at least an example by the teacher. Typically the lectured material and the solved examples were displayed on the blackboard before the activity started. Most of the activities were of this type. Typical examples are 9, 19, 20.

Obviously, there is plenty of room for improvements. However, these activities provided a good opportunity for students to participate and learn mathematics by doing it. Also they gave students the opportunity to contribute in a collaborative effort of their group and class. At the same time they helped the teacher to be more in tune with the class.

Covering the Material. The activities did not present any serious problem in covering the syllabus. My section went at the same pace as the other sections of Math 108. The reason is that they were well integrated in the lectures. For example, the students were doing the concrete examples leading to the general theory to be discussed as an activity instead of the teacher. And that a second application of the theory already discussed was done by the students instead of the teacher. Also, when sometimes the class time was not enough to complete the activity, then the groups were asked to do it after class and return their paper in the next class. Finally there were days with where no activities were given.

Grading. 7 out of the 22 activities given were collected and graded. The scale used was 0-3. the total, 21 points, was entered into the homework, which contributed 50 points in the total of 500 points for the course.

Rules for Activities

- Read material before is discussed in class. (most difficult)
- Work in groups of 2-4.
- Raise hand if group needs help.
- Some of the activities will be collected and graded. Grade will form a part of homework grade. (7 out of 22 were graded, and almost all got perfect score)

Conclusions

- Activities promoted active learning.
- Students helped each other learn. (their way)
- Students helped teacher too!
- Promoted asking questions. (good ones)
- The questions asked by the students provided a valuable and immediate feedback, which resulted to immediate adjustments in teaching.
- Created a sense that everyone learned something in each class.
- Enhanced attendance.
- No sleeping students. (at 8:30 am)

Remarks

- Not easy to make students read material before class.
- Could write better activities.

Activity 1

Today we begin by reviewing integration, and the basic integration techniques of substitution and integration by parts.

A Recall that the antiderivative or indefinite integral of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ i.e. the derivative of $F(x)$ is the original function $f(x)$. This is written $\int f(x)dx = F(x) + C$ where C is a constant. Calculate the following indefinite integrals.

1. $\int x^3 - x^2 + 4dx$

2. $\int e^{2t}dt$

3. $\int t^{-1}dt$

4. $\int y^{1/2}dy$

B Recall that integration by Substitution is based on the **chain rule** for differentiation. For practice compute

5. $\int (x^2 - 3x)^8(2x - 3)dx =$

C Integration by Parts. This method is based on the product rule for differentiation. It uses the formula

$$\int udv = uv - \int vdu.$$

where $dv = v'dx$ and $du = u'dx$. Now use this method to compute the indefinite integral

6 $\int te^{3t}dt =$

Activity 2

Partial Fractions. To compute the indefinite integral $\int \frac{dx}{ax^2 + bx + c}$, in the case that the quadratic in the denominator has real roots r_1, r_2, first we find the numbers A and B such that

$$\frac{1}{ax^2 + bx + c} = \frac{A}{x - r_1} + \frac{B}{x - r_2}.$$

And then we integrate the two simple fractions to compute the desired integral in ters of logarithmic functions. Practice this method by computing the integral

7 $\quad \int \frac{dx}{x^2 + 3x - 4} =$

Activity 3

Today we shall discuss the **definite (Riemann) integral** of a nonnegative function. It is the area of the region under its graph and above the x-axis. To compute this area we use the following simple idea.

The Idea. We cover the region under the graph with very thin non-overlapping rectangles. Each rectangle has its base on the x-axis, and its height is equal to the value of the function at a point in the base. If the rectangles are very thin, the total area they contain will approximate the area of the region under the graph, and by making the rectangles thin enough we can get as good an approximation as we want.

Example. Use this idea to approximate the definite integral of the function shown below.

Activity 4

A. The method of **substitution for indefinite integrals** together with the **Fundamental Theorem of Calculus** give the following method for computing definite integrals

$$\int_a^b f(g(x))g'(x)dx \overset{(u=g(x))}{=} \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a)), \tag{1}$$

where F is an antiderivative of f. Use formula (1) to compute the integral

$\int_1^2 (2x-5)e^{-x^2+5x-6}dx =$

B. Using the **the integration by parts formula for indefinite integrals** and the **fundamental theorem of calculus** we obtain

$$\int_a^b udv = uv|_a^b - \int_a^b vdu, \tag{2}$$

which is the integration by parts formula for definite integrals. Use formula (2) to compute the integral

$\int_1^e t^4 \ln t dt =$

Activity 5

The area between two curves is equal to the area under the **upper curve** **minus** the area under the **lower curve**.

Problem: Compute the area of the region enclosed by the curve $y = 3 - x^2$ and the curve $y = x^2 - 4x + 3$. Sketch the region first.

Activity 6

Today we shall see that the **average value** of a function f over an interval $[a, b]$ is given by the formula:

$$\boxed{\text{average value of } f \text{ over } [a, b] \;=\; \frac{1}{b-a}\int_a^b f(x)dx}.$$

Use this formula to compute the average value of the function

$$f(x) = \frac{1}{x^2 + 5x + 4},\ 0 \leq x \leq 4.$$

Activity 7

Today we shall discuss how we can go from Marginal Function to Total Function by using the Fundamental Theorem of Calculus (FTC):

$$\int_a^b F'(x)dx = F(b) - F(a),$$

which can be stated as the **Fundamental Principle**:

$$\boxed{\text{The definite integral of a rate of change equals the total change.}}$$

Problem. The marginal profit function for producing and selling x units of a certain product is given by $MP = -0.6x + 420$. The company operates at the production level of 500 units. Is it profitable for the company to increase production? If yes, find the production level which yields maximum profit, as well as the additional profit gained by raising production to that level.

Activity 8

Problem. Find the equilibrium quantity, price, the consumer surplus, and the producer surplus for the following demand and supply curves.

$$D(q) = \frac{25}{q+2}, \quad S(q) = q + 2.$$

Activity 9

Problem. Suppose that an amount of money M_0 is deposited in the bank which pays interest at an annual rate r compounded continuously.

a. Find the time T needed for M_0 to double as a function of r.
b. Find T if r is 5%

Activity 10

Problem. An employee of a company is offered a choice between the following two retirement plans. For the first plan the company will open an IRA with the initial amount of $\$5,000$ and thereafter for the next 25 years an income stream of $\$15,000$ will be deposited in the IRA with an annual rate of 8% compounded continuously. For the second plan the company will pay the employee $\$1,100,000$ at the end of 25 years. Which one is the more beneficial plan for the employee.

Activity 11

Problem. Using the trapezoidal rule, estimate the integral

$$\int_0^1 \sqrt{1-x^2}\ dx;\ n=5.$$

Activity 12

Problem. Check that the function $y(x) = a + ce^{kx}$, $a, c, k \in \mathbb{R}$, is a solution to $\dfrac{dy}{dx} = k(y - a)$.

Activity 13

Today we shall consider separable differential equations. A DE is **separable** if the independent and dependent variables can be separated. More precisely if it can be written in the form

$$\boxed{f(y)\frac{dy}{dt} = g(t) \qquad \text{or} \qquad f(y)dy = g(t)dt,} \tag{3}$$

where f and g are known functions. To solve such equations, we integrate both sides and we obtain $\int f(y)\frac{dy}{dt}dt = \int g(t)dt$, which is the same as

$$\int f(y)dy = \int g(t)dt.$$

If these integrals are computed then the DE (1) is solved.

Problem A. Solve the differential equation $\dfrac{dy}{dt} = 4 - y$. Check your answer.

Problem B. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x-1}{y}$, $y(3) = 8$. Check your answer.

Activity 14

A constant solution to a differential equation is called an **equilibrium** solution or **steady-state** solution. An equilibrium solution is called **stable** if any solution arising from a small change in the initial condition tends to the equilibrium solution as the independent variable tends to infinity. It is called **unstable** if a small change in the initial condition gives rise to a solution that moves away from the equilibrium solution.

Problem A. Find the equilibrium solution of each of the differential equations

$$(\mathbf{a})\ \frac{dy}{dt} = y - 20, \qquad\qquad (\mathbf{b})\ \frac{dy}{dt} = 3 - 0.1y,$$

and decide whether they are stable or unstable.

Problem B. Solve these differential equations and then look at the solutions to draw the same conclusions.

Activity 15

To solve a linear system of equations we apply Gaussian elimination. This method consists of performing a sequence of the following

Elementary Operations

1. Interchange equations E_i and E_j.
2. Replace equation E_i by cE_i, where c is a nonzero constant.
3. Replace equation E_i by the combination $E_i + dE_j$, where E_j is another equation and d is a constant.

In practice frequently we combine operations (2) and (3); i.e. we replace an equation E_i by a combination $cE_i + dE_j$, where $c,\ d$ are constants with $c \neq 0$ and E_j is another equation.

Each time we apply an elementary operation to our system we obtain a new system with the same solution as the old one; i.e. we obtain a new system that is **equivalent** to the original one. This is the reason that this method works.

By applying Gaussian elimination we either arrive at an equation of the form $0 = b,$ with $b \neq 0$, which tells us that the system is **inconsistent**; i.e. it has no solution, or we obtain a simple system in the **echelon, or triangular form**, which is solved by back-substitution.

Problem. Apply Gaussian elimination to solve the system

$$
\begin{aligned}
&x_1 - 3x_2 + 4x_3 - 2x_4 = -1\\
&3x_1 - 8x_2 + 10x_3 - 5x_4 = 2\\
&-2x_1 + 7x_2 - 9x_3 + 5x_4 = 6\\
&2x_1 - 5x_2 + 7x_3 - 2x_4 = 4.
\end{aligned}
$$

Activity 16

Problem A. Find the equation of the plane passing through the point $(-4, -2, \frac{9}{2})$ and having slope in x-direction $-\frac{5}{2}$ and slope in y-direction $\frac{7}{2}$.

Problem B. Find the equation of the planes passing through the three points $(1, 0, -1)$, $(0, 1, 3), (1, -1, -2)$.

Activity 17

Given a function $z = f(x, y)$ and z_o a number then the **level curve of height $\mathbf{z}_o$** is the set of all points in the xy-plane satisfying equation $f(x, y) = z_o$. Geometrically, the level curve of height z_o, of a given function $z = f(x, y)$ is obtained by slicing its graph by a horizontal plane at height z_o and then projecting the intersection of the plane with the graph onto the xy-plane. In topographical maps, level curves are drawn with the corresponding altitude to help readers visualize the 3D-picture of the ground surface.

Problem A. Sketch the level curve of the function $f(x, y) = (x-3)^2+(y-5)^2$ of height $z_0 = 4$,and the level curve of the function $f(x, y) = y - x^2$ of height $z_0 = 1$.

If $f(x, y)$ is a function of the two variables (x, y) and if we let only one of the variable be free and the other fixed, then $f(x, y)$ becomes a function of one variable. The ordinary derivative of this function with respect to the free variable (x, or y) is called the **partial derivative** of $f(x, y)$ for that variable.

Problem B. Find the first order partial derivatives the given functions:

1. $f(x, y) = 5 - 8x + 5y$

2. $f(x, y) = x^3y^2 - 2x^2y + y$

3. $f(x, y) = y^3e^{2xy} - x^5$

4. $M(r, t) = 1000e^{rt}$

5. $R(s, t) = \dfrac{5t^2}{s}$

Activity 18

Problem A. Use the first and second derivative tests to find the local minima, maxima, and saddle points of the function $g(x, y) = x^2 - xy + y^2 + 3y - 1$. If the second derivative test is inconclusive, so state.

Problem B. Find the equation of the tangent plane to the graph of the function $z = f(x, y) = x^{1/3}y^{2/3}$ at the point $(1, 8)$, as well as its linear aproximation at the same point. Then use its linear aproximation to estimate $f(1.1, 8.2)$.

Activity 19

Problem. The quantity, Q, of a product manufactured by a company is given by $Q(K, L) = 40K^{0.35}L^{0.65}$, where K is the quantity of capital and L is the quantity of labor used. Capital costs are \$70 per unit, labor costs are \$65 per unit, and the company wants to keep costs for capital and labor combined equal to $\$4,000,000$.Find the combination of capital and labor which gives the maximum output

Activity 20

Problem. Determine which of the given functions is a probability density of a random variable X.

a) $f(x) = \frac{4}{15}x^3, -1 \leq x \leq 2$ b)$f(x) = \frac{3}{4}x(2-x), 0 \leq x \leq 2$

Then compute the probability of $1 < X < 2$.

Activity 21

Problem. Suppose that the amount of milk in a gallon container is a normal random variable X with $\mu = 128.2$ ounces and $\sigma = 0.2$ ounce. Then find the probability that a bottle of milk chosen at random contains more than 128.6 onces.