Math 108 Exam 3 Solutions

November 21, 2002

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1. Using Euler's method with $n = 2$, estimate $y(2)$ where

$$\frac{dy}{dt} = e^{y^2 - 9t}, \qquad y(1) = 3.$$

We have $t_0 = 1, t_1 = 1.5, t_2 = 2.0$ and $y_0 = 3$. The first step of Euler's method gives: $y_1 = 3 + (0.5)e^{3^2-(9)(1)} = 3.5$. The second step gives the answer: $y_2 = 3.5 + (0.5)e^{(3.5)^2-9(1.5)} = 3.5 + 0.5e^{-1.25}$.

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2. Which of the following is the second order Taylor polynomial approximating $f(x) = \ln(2x+1)$ at $x = 0$?

The first derivative is $f^{(1)}(x) = \frac{2}{2x+1}$ (remember- *chain rule!*). The second derivative is $f^{(2)}(x) = \frac{-4}{(2x+1)^2}$. Evaluating at $a = 0$ gives: $f^{(0)}(0) = 0$, $f^{(1)}(0) = 2$, $f^{(2)}(0) = -4$. So the Taylor polynomial is $P_2(x) = 0 + \frac{2}{1!}(x-0) + \frac{-4}{2!}(x-0)^2 = 2x - 2x^2$.

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3. Use a Taylor polynomial of degree 3 about $x = 1$ to approximate

$$\int_1^2 \ln(x)\, dx$$

We have $f^{(1)}(x) = \frac{1}{x}$, $f^{(2)}(x) = \frac{-1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$. So the Taylor polynomial is $P_3(x) = \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$. The approximating integral is:

$$\int_1^2 \left[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right] dx = \left[\frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12}\right|_1^2 = (\frac{1}{2} - \frac{1}{6} + \frac{1}{12}) - (0) = \frac{5}{12}$$

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4. Consider the series

$$\sum_{k=0}^{\infty} \left(\frac{5}{4}\right)^k.$$

Which of the following statements is true?

The geometric series $1 + x + x^2 + x^3 + \cdots$ converges if and only if $|x| < 1$. This is the geometric series evaluated at $x = \frac{5}{4}$, so it diverges.

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5. The test for a particular disease is positive 96% of the time when the disease is present. The test is also (false) positive 2% of the time when the disease is not present. If 0.2% of the population has the disease, what is the probability that a person who tests positive actually has the disease? (choose the closest answer).

This is a problem for Bayes' Formula. Let $E =$"The test is positive" and $F =$"The disease is present". We are given $P(E|F) = 0.96$, $P(E|F') = 0.02$ and $P(F) = 0.002$. Applying Bayes' formula gives

$$P(F|E) = \frac{(0.96)(0.002)}{(0.96)(0.002) + (0.02)(0.998)} \approx 0.0878.$$

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6. A fair coin is flipped 4 times. What is the probability that it comes up "heads" exactly two times?

The sample space for this experiment is $S =${HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}. The event we seek is $E =${HHTT, HTHT, HTTH, THHT, THTH, TTHH}. All the outcomes are equally likely, so $P(E) = \frac{6}{16} = \frac{3}{8}$.

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7. If E and F are two events satisfying $P(E') = 0.6$, $P(F) = 0.6$ and $P(E \cup F) = 0.7$, find $P(E \cap F)$.

First notice that $P(E) = 1.0 - 0.6 = 0.4$. Then $P(E \cap F) = P(E) + P(F) - P(E \cup F) = 0.4 + 0.6 - 0.7 = 0.3$.

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8. Suppose two cards are drawn in succession from a standard deck. What is the probability that both are of the same suit?

We can reason this problem out in the following way: Whatever was drawn the first time, there are 12 cards left of that suit and 51 cards left in total. So the probability of drawing a second card from the same suit as the first is $\frac{12}{51}$.

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9. A monkey is randomly hitting keys on a typewriter with 26 keys (i.e., 'A', 'B',...,'Z'). If he types exactly 7 characters independently, what is the probability that it will read "OTHELLO"?

We are told that the characters are typed independently. The probability that the first character is an 'O' is $\frac{1}{26}$. The probability that the second character is a 'T' is $\frac{1}{26}$, and so on. Since these events are independent, we simply multiply the probabilities: $\left(\frac{1}{26}\right)^7$.

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10. What is the value of the following series?

$$1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^3 + \cdots$$

This is simply the geometric series $\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$. Since the argument $\frac{3}{5}$ has absolute value less than 1, the series converges to

$$\frac{1}{1 - \frac{3}{5}} = \frac{1}{\frac{2}{5}} = \frac{5}{2}.$$

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11. A $\frac{3}{4}$ pint cup of coffee is sitting on my desk evaporating. Suppose that 1/3 of a pint evaporates on the first day, 1/9 of a pint evaporates on the second day, 1/27 of a pint evaporates on the third day, and so on. If this process continued forever in this way, how much coffee would eventually remain in my cup?

The amount of coffee that will evaporate is

$$\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots = -1 + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = -1 + \frac{1}{1 - \frac{1}{3}} = \frac{1}{2}.$$

We started with $\frac{3}{4}$ of a pint, so $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ pint remains.

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12. Consider events E and F satisfying $P(E|F) = 0.3$, $P(E) = 0.1$, $P(F) = 0.6$. What is $P(E \cap F)$?

$P(E|F) = \frac{P(E \cap F)}{P(F)}$, so $0.3 = \frac{P(E \cap F)}{0.6}$. Multiplying both sides by 0.6 gives $P(E \cap F) = (0.3)(0.6) = 0.18$.

13. Which of the following is the Taylor series approximating e^{x+2} about $x = 1$?

If $f(x) = e^{x+2}$, then all of its derivatives are equal to e^{x+2}. Thus, $f^{(k)}(1) = e^3$ for all $k \geq 0$. So the Taylor series is given by

$$\frac{e^3}{0!} + \frac{e^3}{1!}(x-1) + \frac{e^3}{2!}(x-1)^2 + \frac{e^3}{3!}(x-1)^3 + \cdots = \sum_{k=0}^{\infty} \frac{e^3(x-1)^k}{k!} = e^3 \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}.$$

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14. (Two parts)
(i) Use Euler's method with $n = 2$ to approximate $y(1)$ where

$$\frac{dy}{dt} = -2y, \qquad y(0) = 1.$$

We have $y_0 = 1$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1.0$. We first compute $y_1 = 1 + (0.5)(-2 \cdot 1) = 0$. Then $y_2 = 0 + (0.5)(-2 \cdot 0) = 0$.

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15. An experiment consists of rolling two dice and observing the numbers on the sides facing up.
(a) List the sample space for this experiment.
(b) Find the probability that the smaller of the two numbers is at least 4.

The sample space is $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \ldots, (6,6)\}$. (There are 36 outcomes in the sample space). Let E be the event "The smaller of the two numbers is at least 4". Looking at the sample space, we conclude that $E =\{$ (4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5), (6,6)$\}$. Since all 36 outcomes are equally likely and E consists of 9 outcomes, $P(E) = \frac{9}{36} = \frac{1}{4}$.

Another solution to part (b) is as follows: Let E denote the event "the first die is 4, 5, or 6", and F be the event "the second die is a 4, 5, or 6". Since these events are independent, we can compute: $P(E \cap F) = P(E)P(F) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$.