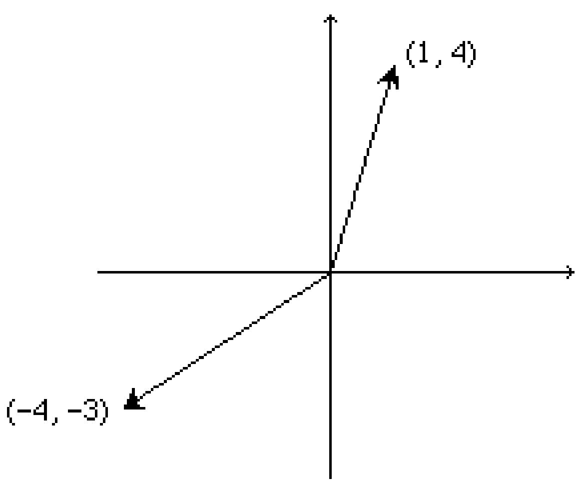
REVIEW FINAL EXAM

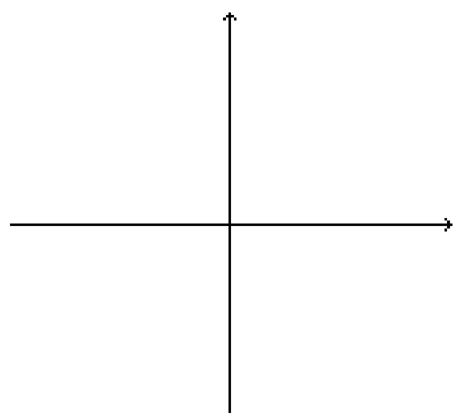
1. Compute the derivative of $f(x) = (x^2 - 1)^{\frac{1}{3}}$. After having done so, find all the critical points. Determine precisely the portions of the x-axis for which the function $f(x)$ is increasing and decreasing. Use this information to find the local maxima and minima.

2. Consider the function $f(x) = \frac{2}{5} x^{\frac{5}{2}} + 32 x^{-\frac{1}{2}}$ for $-5 \le x \le 5$. Find all the critical numbers for this function. Find the number or numbers for which this function attains its maximum and minimum values. What is the maximum value of this function and what is its minimum value ?

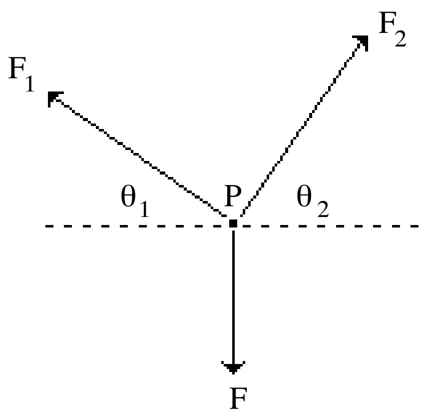
3. Find the critical numbers of the function $f(x) = 3x^{\frac{1}{3}}(x - 8)$.

4. The two vectors in the x-y plane below represent forces. Draw in the horizontal and vertical components of each of the two vectors. Specify their endpoints. Put into the x-y plane on the right the vector that represents the combined effect of the two forces. Again specify the endpoint.





5. The three forces below are in equilibrium. They all act on the point P. The dotted line is horizontal and F acts vertically. If F_1 has a magnitude of 10 pounds, and if the angles θ_1 and θ_2 are 30^o and 60^o respectively, determine the magnitudes of F_2 and F.



6. You are given the following parameters for a suspension bridge. The center span is equal to 4200 feet. The dead load supported by the two main cables over the center span is 21,300 pounds per foot and the live load capacity of 4000 pounds per foot. The sag in each of the two main cables is 470 feet. Compute the maximal tension T_d in one of the main cables over the center span and the angle α that it makes with the horizontal at the point at which it meets the tower. What compression does this cable produce in the tower ? What must the horizontal component of the tension of each of the cables over the side span be (at the point at which it meets the tower) so that the bridge is stable ?

7. The Tacoma Narrows Bridge was completely rebuilt in the years 1948 to 1950. The data for the new bridge is as follows. It has a total length of 5,000 feet and a center span of 2,800 feet. Its two main cables support a single deck that carries 4 lanes of automobile traffic. The dead load is 8,680 pounds per foot. Assume that it is designed for a live load capacity of 4,000 pounds per foot and that the sag in the sag in the cable over the center span is 280 feet. (Based on data supplied by the Washington State Department of Transportation.) Consider one of the main cables over the center span at a point where it meets one of the towers. Compute the tension of one of the cable at that point and compute the angle that the cable makes with the horizontal at that point. Determine the compression that this cable generates in the tower.

8. a) Solve $5^{(x^2-1)} = C$ for x.

   b) Express $f(x) = \log_6 x$ in terms of ln. Use this to compute the derivative of $f(x)$.

   c) Express $g(x) = 7^x$ in terms of the function e^x. Use this to compute the derivative of $g(x)$.

9. Solve $\log_5 (4x^2 - 11) = 2$ for x.

10. Compute the derivatives of the following functions.

    a) $f(x) = \ln x$ b) $g(x) = \ln (x^4 + 3x^2)$

c) $f(x) = e^{x}$ d) $g(x) = e^{(x^2-1)}$

11. Compute the derivative of $f(x) = e^{\sqrt{x^2+1}}$. After having done so, explain why 0 is the only critical number. Does the function have a local maximum or minimum or neither at 0 ? Explain why. What is the (absolute) minimum value of the function $f(x)$?

12. One of the waste products of a nuclear explosion is the radio-active isotope strontium–90. This isotope which behavers chemically like calcium, has a half-life of (about) 25 years. If 20 milligrams of the isotope are present in a sample now, find

 i. how much will remain in 15 years
 ii. in how many years will only 5 milligrams remain ?

13. A radiation counter shows that a certain radioactive substance disintegrates at a rate of 5.2×10^{15} atoms per second at a certain time and at a rate of 3.7×10^{12} atoms 5 minutes later. Determine the half-life of this radioactive substance.

14. Rutherford studied the radium isotope ${}^{224}_{88}\mathrm{Ra}$. In one experiment he made the following measurements for the decay rate $y'(t)$ of the sample he was testing: at $t = 1$ day, $\frac{y'(t)}{y'(0)} = 0.88$; at $t = 2$ days, $\frac{y'(t)}{y'(0)} = 0.72$; at $t = 4$ days, $\frac{y'(t)}{y'(0)} = 0.53$; at $t = 7$ days, $\frac{y'(t)}{y'(0)} = 0.295$; at $t = 8$ days, $\frac{y'(t)}{y'(0)} = 0.252$; at $t = 11$ days, $\frac{y'(t)}{y'(0)} = 0.152$; at $t = 13$ days, $\frac{y'(t)}{y'(0)} = 0.111$. What value λ for the disintegration constant of radium–224 could Rutherford deduce from this data ?

15. A fragment of a mineral grain is found to contain 305 parts per million rubidium–87 and 4.67 parts per million strontium–87. Given that the half-life of rubidium–87 is 4.7×10^{10} years, determine the age of the fragment.

16. A wood fragment tested with an accelerator mass spectrometer and is found to contain carbon-14 atoms to stable carbon atoms in a ratio of 1 to 1.6174×10^{12}. Compute the age of the fragment. Assume that at the time the metabolic processes in the wood stopped, the equilibrium ratio of radioactive carbon to stable carbon was the same as it is today.

17. Fragments of skeletons unearthed near the town of Arella, Pennsylvania gave evidence of a civilization that existed in this area from around 14,300 to 15,000 years ago. Estimate the ratio of carbon–14 atoms to stable carbon atoms in the fragments that warranted this conclusion.

18. Suppose that a colony of bacteria is growing exponentially and $y(t)$ = according to the phase. If the number of bacteria increases from 5000 to 7000 in 12 hours, find the doubling time.

19. A bacteria culture is in its exponential growth phase. It starts with 10,500 bacteria at time $t = 0$. Two hours later there are 23,000 bacteria.

   a) Express the number $y(t)$ of bacteria at any time $t \geq 0$ in terms of an exponential function.
   b) What is the size of the population after 5 hours ?
   c) At what time will the population reach 130,000 ?

19. (Study this from the notes. It occurs in the development of the logistics equation.) Use the "reverse the common denominator method" to express the indefinite integral $\int \frac{1}{(x-2)(x-3)}\,dx$ as a sum of two simpler indefinite integrals. Use this to solve the integral.

20. The statistics for the population of a certain country shows the following:

8.45 million at the beginning of 1975 and that it increased at a rate of 0.24 million per yr. in 1975
12.82 million at the beginning of 1980 and that it increased at a rate of 0.35 million per yr. in 1980
17.83 million at the beginning of 1985 and that it increased at a rate of 0.47 million per yr. in 1985
24.74 million at the beginning of 1990 and that it increased at a rate of 0.63 million per yr. in 1990
30.16 million at the beginning of 1995 and that it increased at a rate of 0.72 million per yr. in 1995

Let $t = 0$ correspond to the year 1975. Let $y(t)$ be the population of this country at any time $t \geq 0$ in years.

   a) Show that the population of this country satisfies the basic assumption of the logistics model and determine the parameters y_0, k, and M.
   b) What is the limit on the population of this country ?
   c) After inserting all relevant constants, $y(t)$ =
   d) Assuming that the growth pattern observed in the period from 1975 to 1995 continues, what would you expect the country's population to be in the year 2020.

21. An amount of \$5,000 is invested in an account that pays an interest rate of $r = 0.08$. What will the investment be worth after 7 years, if interest is

   i. compounded annually ?   ii. compounded semi-annually ?

iii. compounded quarterly ? iv. compounded monthly ?
v. compounded daily ? vi. compounded continuously ?

22. An amount of A_0 dollars is deposited in an account that pays interest at an annual rate of 0.07 compounded continuously. In how many years will the account double to $2A_0$ dollars ?

23. Your aunt has been paying $20 per month into an account earning interest at an annual rate of 0.06 starting on the day you were born. She will give you all the money that has accumulated in this account on your 21st birthday. How much will you get?

24. Suppose that an investment plan is set up with a bank that calls for a monthly investment of $250. Suppose that the plan earns interest at an annual rate of $r = 0.07$ compounded monthly. How much money will be in the account 10 years after it is started ? How much money will it have after 15 years ? How much money will this account have after 10 years if interest is paid at an annual rate of $r = 0.08$? How much will it have after 15 years at $r = 0.08$?

25. Arrangements have been made with a bank for a mortgage of $100,000 at an annual interest of 7% compounded monthly. Suppose that it is taken out now and that it is to be paid back in monthly payments over the next 30 years. What will the monthly payments be ? Hint : Let B be the monthly payment. From the bank's point of view the repayment can be viewed as a monthly annuity which pays it B dollars per month for $30 \times 12 = 360$ months. If the bank is willing to exchange the loan for this stream of payments, then this loan of $100,000 must be the present value PV of this income stream.

26. The purchaser of a home obtains a $120,000 mortgage at an annual interest rate of 7%. The agreement with the bank calls for the mortgage to be paid off with fixed monthly payments over 25 years. How much are the monthly payments?

27. Suppose that the 30 year mortagage of $120,000 of Exercise 25 is to be paid back at the annual interest of 8% (still compounded monthly). What will the monthly payments be then ?

28. A younger colleague of the professor whom we met in the text turned 40 in May of 1994 and has (since that time) been paying $1,000 per month into an account that pays at an annual rate of $r = 0.05$ compounded monthly. How much money will she have in this account when she retires at the age of 70 in May of the year 2024 ? She has decided to use this account to provide her with monthly annuity payments for the first 15 years of her retirement. What monthly payments will she receive from this account if it pays interest at an annual rate of $r = 0.06$ compounded monthly?

The companies ATT, IBM, and Texaco, all issued $1000 bonds in the 1990s. ATT issued 250,000 bonds on August 1, 1967; IBM issued 1,250,000 bonds on June 15, 1993; and Texaco issued 200,000 bonds on Feb 15, 1991. The coupon payments occur on February 1 and August 1 for the ATT bond, on June 15 and December 15 for the IBM bond, and on

February 15 and August 15 for the Texaco bond. The bond page of February 27th, 1996 provides the following information about three bonds.

ATT 6s00	6.0	203	$100\frac{1}{8}$	$-\frac{1}{4}$
IBM $6\frac{3}{8}$ 00	6.3	125	102	...
Texaco $8\frac{1}{2}$ 03	7.6	5	112	–3

The table lists, the amount of the semi-annual coupon payment, the year of maturity of the bond, the number of bonds traded on February 27th and the closing price, and the closing price the day before.

29. In reference to each of the bonds: How many coupon payments remain ? Write a formula for the present value on February 27th. Determine the present value for an interest rate of 0.06. Estimate the interest rate on which the closing price was based.

30. The CPI was 153.2 in September of 1995. Estimate the CPI in September of the year 2015 if the annual inflation rate over the twenty years in question will average 4% per year. Given that a Notre Dame education costs about \$25,000 per year in the academic year 1995/96, estimate the yearly cost of a Notre Dame education in the academic year 2015/2016.

31. The CPI (base year 1987) was 31.6 in July 1965 and 152.5 in July 1995. What was the average rate of inflation over this 30 year period ? The presidency of Jimmy Carter (1976 to 1980) was plagued by high inflation (brought about to a large extent by increases in the price of oil). Estimate the average rate of inflation for the 2 year period from December 1978 to December 1980. Use the fact that the CPI was 67.7 in December 1978 and 86.3 in December 1980.

32. The short run supply and demand functions for a given product in a given market are both linear. At a price of 7 dollars, the demand and supply are both 220,000 units per month. At this price the price elasticity of supply is 0.20 and the price elasticity of demand is –0.12.

    a) Determine the supply and demand functions for the product.
    b) A group of suppliers gets together and (for the purpose of driving up the price) decides to cut production by 40,000 units. Assuming that the short run demand situation remains the same, estimate the new equilibrium price that market forces will determine.

33. Continue to consider the situation of Section 3B. Develop a linear demand function D for the world's oil under the assumption that at 4 dollars per barrel the demand is 18 b. b. per year and the price elasticity of demand is –0.08. Given an OPEC supply cut to 10 b. b. per year (insted of 9), what would the market clearing price p^* have been under these assumptions ?

All formulas will be supplied. For example:

$$T_0 = T_x \cos\theta \qquad wx = T_x \sin\theta \qquad \frac{wx}{T_0} = \tan\theta \qquad f(x) = \frac{w}{2T_0}\,x^2$$

$$T_x = w\sqrt{\frac{1}{4}\frac{d^4}{s^2} + x^2} \qquad \tan\alpha = f\,'(d) = \frac{2s}{d^2} \quad d = \frac{2s}{d}$$

$$y(t) = y_0\,e^{-\lambda t} \qquad y'(t) = -\lambda y(t) \qquad 1\,\mathrm{g} = \frac{1}{m}(6.02 \infty 10^{23})$$

$$t = \frac{1}{\lambda}\ln\left(\frac{z(t)}{y(t)} + 1\right) \qquad t = (1.89 \times 10^9)\ln\left(\frac{9.07z(t)}{y(t)} + 1\right)$$

$$t = (8.26 \times 10^3)\ln\left(r_0\frac{k}{y(t)}\right) \qquad r_0 = \frac{1}{6.463 \infty 10^{11}}$$

$$\frac{y'}{y} = k - cy \qquad a = ck^{-1} \qquad M = \frac{1}{a} \qquad M = kc^{-1} \qquad \frac{y'}{y} = k - \frac{k}{M}\,y = k(1 - \frac{y}{M})$$

$$y(t) = \frac{Me^{kt}}{C + e^{kt}} \quad \text{or} \quad y(t) = \frac{M}{1 + Ce^{-kt}} \quad \text{where } C = \frac{M}{y_0} - 1$$

$$A_p = A_0(1 + \frac{r}{n})^p \qquad A(t) = A_{nt} = A_0(1 + \frac{r}{n})^{nt} \qquad A(t) = A_0e^{rt}$$

$$S_p = \frac{12}{r}A_0\,(1 + \frac{r}{12})\left((1 + \frac{r}{12})^p - 1\right) \qquad PV_p = \frac{12B}{r}\left[1 - (1 + \frac{r}{12})^{-p}\right]$$

$$PV_p = \frac{2C}{r}\left[1 - (1 + \frac{r}{2})^{-p}\right] \qquad p(t) = p_0\,e^{kt}$$

$$e_D(p) = p\,\frac{D'(p)}{D(p)} \qquad e_S(p) = p\,\frac{S'(p)}{S(p)}$$