Lecture 1

Math 112 Beginning Logic
MWF 1:55-2:45
282 Galvin

Instructor: Julia Knight
Office: 233 MCC
Hours: MWF 3-4 (tentatively)

Roll:

Text: E. J. Lemmon, Beginning Logic (with supplementary material later)
The book sounds as if it was written for British schoolboys around the turn of the century, to sit beside leather-bound books on rhetoric and Latin grammar. The tone may be off-putting, but the book has the right level. It does not assume any mathematical background.

Coursework and Grading

In-class exams., three such, 15% each
Final, 30% (according to university rules)
Homework, 15% (done in groups, more about that in a moment)
Paper, talk 10% (on material beyond text, more about that later)

Dates

Exam. I: Feb. 7
Exam. II: March 5 (Wed. before Spring break)
Papers: due March 26 (Wed. before Easter)
Exam. III: April 11
Talks: April 21-25
Final: May 5, 8:00-10:00 (first day of finals)

I'll assign homework regularly. Some of the homework will be challenging, constructing proofs. I want you to work in groups. I think that you will find that working together, talking about the material helps you understand it better. Moreover, you will almost certainly be working in groups when you take a regular job.

Assign groups of 3-4.

The first assignment will be to read through p. 8 in the text, and to talk with the members of your group about why you signed up for the course and what you hope to get out of it. I would like each group to turn in a short (< 1 page) statement, jointly written, summarizing this discussion. (Your group may designate a scribe, or each write a paragraph, but all should read and approve the whole before you turn it in on Friday. If you use a computer, it will be easy to fold separate paragraphs together, or make changes.)

In the remaining time, I'd like to say a little about what logic is. What do we do in logic ?

I. <u>Analyze arguments</u>

We say what is a "sound" argument.

Argument starts with some premises (assumptions) and arrives at a conclusion. These statements may involve mathematical objects such as numbers, or they may involve family relationships or historical facts.

In logic, we look at the <u>form</u> of the argument. We try to say when the form guarantees that anyone who accepts the premises has to accept the conclusion.

<u>Example 1</u> (famous)

Premises: All men are mortal.

Socrates is a man.

Conclusion: Socrates is mortal.

<u>Example 2</u>

Premises: For all natural numbers n, $1+2+3+...+n = \frac{n(n+1)}{2}$.

7 is a natural number

Conclusion: $1+2+3+4+5+6+7 = \frac{7 \cdot 8}{2}$ (= 28)

Examples 1 and 2 have the same form (syllogism). I think you believe both to be sound.

The next example is more complicated. I won't tell you whether it is sound.

<u>Example 3</u>

Premises: If the river floods, then our entire wheat crop will be destroyed.

The river will flood if there is an early thaw.

In any case, there will be heavy rains later in the summer.

Conclusion: If there is an early thaw, our entire community will be bankrupt unless there are heavy rains later in the summer.

Part of the difficulty in analyzing this argument is the length of the sentences. It would be easier to analyze something short, of the same form. Another difficulty is that some words may be hard to interpret. This leads to something else we do in logic.

II. Develop formal languages

Formal languages let us examine the form of an argument, or of a single statement. Translating an argument from English into an apropriate formal language makes the form transparent. We remove ambiguities.

I and II are the parts of logic that we shall concentrate on for most of the course. We shall discuss some formal languages, and formal system of proof. Homework will involve translating from English into an appropriate formal language, and analyzing and constructing proofs in formal system.

Students who took similar course in the past, arts and letters students, felt that this kind of exercise should improve their ability to analyze complex material, and their ability write to make a point.

There are other kinds of problems, other areas of logic.

III. Describe what is (or is not) computable

What can be done by a machine ? If there is an algorithm (method which could be turned into a computer program) for doing something, you could show this by describing the algorithm. For example, there is an algorithm for deciding whether a natural number n is a prime. Divide n by 2, 3, 4, ..., n-1. If you don't come to a number which divides n, then n is prime. There is an algorithm for deciding whether a polynomial equation in many variables has a solution in real numbers. No such algorithm for solution in integers.

Example of polynomial equation: $7x^2y^3z^5 - 13xyz^2 + 5x^{16}z - 2xz^4 = 0$

American logician, Julia Robinson, reduced problem to something else. Finished by Russian logician, Yuri Matijasevich.

IV. Develop theory of sets

Set theory serves as foundation for mathematics. Discuss the sizes of sets, both finite and infinite. What makes one infinite set larger than another ?

V. Analyze expressibility, definability

Consider the relation between formal languages and real situations, or mathematical structures. What can you say, in a particular setting, using a particular language ? Using a natural language for talking about numbers, in real numbers cannot define or say anything very complicated. In rationals, situation is quite different.

How old is logic ? How did it develop ? Certainly, there were proofs, rules for such, tied to development of geometry in Egypt. Fact that Nile flooded and removed markers between plots of land led to squabbles that could only be settled by sound reasoning from universally accepted geometric premises and observations on distant landmarks.

Formal languages are more recent. Some of the motivation for study of formal languages came from paradoxes. These also motivated more systematic study of sets.

Examples of Paradoxes

1. Liar paradox (ancient Greece)

A man says "I am lying".

If the man is lying, then he is telling the truth, while if he is telling the truth, then he is lying.

2. Russell's paradox (around 1900)

Intuitively, we think of a set as any collection of objects (set of people in South Bend, set of integers).

Let A be the set of all sets X such that X does not belong to itself.

If A belongs to A, then A does not belong to A, and if A does not belong to A, then A belongs to A.

3. Richard's Paradox

Some English phrases denote real numbers. For example, "the ratio between the circumference and the diameter of a circle" denotes the number π. We can order the phrases, shortest first, those of same length lexicographically (as in dictionary). Let $r_1,r_2,r_3,...$ be the numbers denoted by 1st, 2nd, 3rd phrase, in order. Now let r be the real number with 1 in the nth decimal place if r_n does not have 1 there and 2 otherwise.

r may be .12211122122211.... Then $r \neq r_n$ for all n, but r is described by an English phrase.

Logicians found ways around the paradoxes by developing formal languages, and formalizing set theory. The ideas behind the paradoxes were used in sound arguments.

1. The idea of Russell's paradox was used to show that the set of subsets of a given set is always larger than the set itself.

2. The idea of the Liar paradox was used to show that there is no algorithm for deci↔ding which sentences are true of the natural numbers with + and ·.

3. The idea of Richard's paradox was used to show that the set of points on a line is larger than the set of integers.

The paper that you write will be on material from logic, not from the text, something you decide to learn about and explain, possibly something on non-computable functions, or set size. I'll say more about this later on.

Homework #1: Meet with your group (assigned groups of size 3-4). Say why you signed signed up for course and what you hope to get out of it.

Lecture 2

Collect statements from groups.

Now, begin developing first formal languages.

Author introduces propositional languages informally in Chapter 1, also introduces rules of proof there. Describes symbols and rules for writing propositional statements more formally in Chapter 2.

Today, we will begin with the more formal description of propositional symbols and statements. Then we will do some translation, see how formal statements match up with statements in ordinary English.

The material is partly from Chapter 1 and partly from Chapter 2.

Recommend that you take notes, then skim Chapters 1 and 2 over the weekend.

We start with symbols for some basic sentences, and combine them using symbols for "and", "or", "implies", "if and only if", and "not".

Symbols

propositional variables

P, Q, etc.

logical connectives

& (and)

∨ (or)

¬ (not)

→ (if ___ then___, or implies)

↔ (if and only if)

parentheses (,), for clarity

The class of well formed formulas is defined as follows:

1. A propositional variable P, Q, etc., is a well formed formula.

2. If F is a well formed formula, then so is ¬F.

3. If F and G are well formed formulas, then so are (F & G), (F ∨ G), (F → G), (F ↔ G).

4. Nothing is a formula unless it can be gotten from by finitely many applications of the rules 1, 2, and 3.

Example 1: (((P & ¬Q) ∨ R) ↔ ((R & ¬Q) → ¬P))

We can show that this is a well formed formula by giving a finite list, each an application of rule 1, or the result of applying rule 2 or rule 3 to earlier items on the list. applying derivation as follows:

1. P (rule 1),
2. Q (rule 1),
3. R (rule 1),
4. ¬Q (rule 2 applied to 2),
5. ¬P (rule 2 applied to 1),
6. (P & ¬Q) (rule 3 applied to 1,4)
7. (R & ¬Q) (rule 3 applied to 3,4),
8. ((P & ¬Q) ∨ R) (rule 3 applied to 6,3),
9. ((R & ¬Q) → ¬P) (rule 3 applied to 7,5),
10. (((P & ¬Q) ∨ R) ↔ ((R & ¬Q) → ¬P)) (rule 3 applied to 8,9)

Later, we may drop parentheses where we can do so with no loss in clarity. For now, let's put them all in.

Example 2: P¬Q→(

This is clearly not a well formed formula. It is a finite sequence of symbols. Author calls it a formula. I may slip up and say "formula" for "well formed formula".

You may think about whether you could take an arbitrary finite sequence of symbols and determine whether it is a well formed formula.

Let us do some translation.

Example 1: Let P stand for "It is snowing", let Q stand for "the temperature is below 0", and let R stand for "classes are canceled".

1. (P ∨ Q)
It is snowing or the temperature is below 0. (Allow both.)

2. (P → R)
If it snows, then classes are canceled.
Classes are canceled if it snows.

3. $(P \leftrightarrow Q)$
Classes are canceled if and only if it snows.
If it snows, classes are canceled, and not otherwise.
Classes are canceled just in case it snows.

4. Classes are canceled only if the temperature is below 0.
If classes are canceled, then it is snowing.
$(R \rightarrow Q)$

4. Classes are canceled if either it is snowing or the temperature is below 0.
If it is snowing, or if the temperature is below 0, then classes are canceled.
$((P \vee Q) \rightarrow R)$

5. If it is snowing and (in addition) the temperature is below 0, then classes are canceled.
$((P \& Q) \rightarrow R)$

6. $\neg P$
It is not snowing

7. $\neg(P \vee Q)$
It is not either snowing or below 0. It is neither snowing nor below 0.

The next examples involve the words "but" and "unless". Sometimes in English, we use "but" for "and". "Unless" is roughly the same as "or".

Example 2:

Let P be "It is Friday", and let Q be "The sun is shining."

1. $(P \& Q)$

It is Friday and the sun is shining.

2. It is Friday but the sun is not shining.

$(P \& \neg Q)$

3. The sun shines unless it is Friday.

We may re-phrase this, saying one of the following
The sun shines or it is Friday.
If the sun is not shining, then it must be Friday.

$(Q \vee P)$ or $(\neg Q \rightarrow P)$

4. The sun does not shine unless it is Friday.

We may re-phrase this, saying one of the following:
The sun is not shining or it is Friday.
If the sun is shining then it must be Friday.

$(\neg Q \vee P)$ or $(Q \rightarrow P)$

<u>Example 3</u>.

1. Tom embezzled money from ABAG and if he is caught he will go to jail.

Let P stand for "Tom embezzled money from ABAG". Let Q stand for "He is caught." Let R stand for "He goes to jail."

Which is the correct translation ?

$(P \,\&\, (Q \rightarrow R))$, or
$((P \,\&\, Q) \rightarrow R)$

What is the difference ?

First says that P holds and, in addition, $(Q \rightarrow R)$ holds.
Second says that if P and Q both hold, then so does R.

2. He will go to jail if he is caught.
$(Q \rightarrow R)$

3. Tom will go to jail just in case he is caught.
$(Q \leftrightarrow R)$

Homework #2:

A. Show that $((P \& Q) \rightarrow \neg\neg R)$ is a well formed formula.

Give two different sequences of steps applying the rules 1, 2, 3 to arrive at the given formula.

B. Translate each of the following from symbols to English, letting A stand for "Alice minds the baby," letting B stand for "The baby cries," and letting C stand for "The Cheshire cat makes trouble."

1. $(A \& (B \& C))$

2. $((A \& B) \& C)$

3. $(\neg(A \& B) \& C)$

4. $(A \& \neg(B \& C))$

5. $(A \& (B \vee C))$

6. $((A \rightarrow B) \& C)$

7. $(\neg(A \rightarrow B) \& C)$

8. $(A \rightarrow \neg(B \& C))$

9. $(A \rightarrow (B \vee C))$

10. $((A \rightarrow B) \vee C)$

C. Consider the following argument.

Premises:
1. If the river floods, then the wheat crop will be destroyed.
2. The river will flood if there is an early thaw.
3. In any case, there will be heavy rains later in the summer.

Conclusion: If there is an early thaw, then our community will be bankrupt unless there are heavy rains later in the summer.

Say what choice of P, Q, R, S, and T could be used to give this argument the form below.

Premises:
1. $(P \rightarrow Q)$
2. $(R \rightarrow P)$
3. S

Conclusion: $(R \rightarrow \neg(S \rightarrow T))$

Lecture 3

Collect homework.

Statements on what you expect to get out of course were interesting. I'd like to keep these if I may. You will get back other homework papers.

On the homework for today, I urged you to work together, but asked you to turn in individual papers. Next assignment will be for groups.

How did homework go ?

We will do more translation as we go along.

Formal system for arguments

Some people believe that writing arguments in a formal system is mental exercise that gets the mind in shape for strenuous thinking of all kinds. Whether or not you believe this, you will see that arguments written this way can be checked, even by a machine.

Rules in the system may look daunting at first, but we will spend enough time on this that you will all be able to write arguments in this system.

You should read Chapters 1 and 2, skimming first, but returning to certain parts repeatedly until they make sense.

An argument for F from set A is a finite sequence of statements $F_1,\ldots,F_n$, ending with $F_n = F$, such that for each i, F_i is obtained by applying some rule of inference. We decorate each statement, keeping track of assumptions and rule used.

We write A |- F if we have argument in which the assumptions used for F come from A.

As rules are introduced, see if you believe that arguments using them are going to be sound.

First rule is often used.

1. Rule of Assumption (A). This rule allows us to write down a statement so that we can use it later.

F |- F

This rule is good in the sense that if you believe F, then you have to believe F.

Second rule is non-trivial, has a Latin name.

2. Modus Ponens, or Modus Ponendo Ponens, (MPP). This rule says that from F and (F → G), we can conclude G.

F, (F → G) |- G

Is this a good rule ? If you believe F and (F → G), would you have to believe G ?

There are more rules, but let us look at argument using just these two. We will introduce the decorations, keeping track of assumptions and rules used.

Example 1: Here is argument showing that P, (P → Q), (Q → R), (R → S) |- S.

1	1. P	A
2	2. (P → Q)	A
1,2	3. Q	1,2 MPP
4	4. (Q → R)	A
1,2,4	5. R	3,4 MPP
6	6. (R → S)	A
1,2,4,6	7. S	5,6 MPP

Look at the decorations. We number the steps 1-7. At the left, we keep track of all of the assumptions behind the given step. At the right, we keep track of the rule being applied, and the previous steps, if any, used for the current step. The numbers at the left all appear first with rule A.

The third rule also has a Latin name.

3. Modus Tollens, or Modus Tollendo Tollens (MTT).

This rule says that if we have (F → G) and ¬G, we can conclude ¬F.

Is this a good rule ? If you believe (F → G) and ¬G, must you believe ¬F ?

Example 2: Here is argument showing that ¬R, (P → Q), (Q → R) |- ¬P

1	1. (Q → R)	A
2	2. ¬R	A
1,2	3. ¬Q	1,2 MTT
4	4. (P → Q)	A
1,2,4	5. ¬P	4,3 MTT

The Fourth rule is really two rules in one.

4. Double Negation (DN).

First, if we have ¬¬F, we can conclude F.

¬¬F |- F

Second, if we have F, we can conclude ¬¬F.

F |- ¬¬F

Are these good rules ? If you believe ¬¬F, must you believe F ?

If you believe F must you believe ¬¬F ?

<u>Example 3</u>: Here is argument showing that (P → ¬Q), Q |- ¬P

1	1. (P → ¬Q)	A
2	2. Q	A
2	3. ¬¬Q	2 DN
1,2	4. ¬P	1,2 MTT

<u>Example 4</u>: Here is argument showing that (¬P → Q), ¬Q |- P

1	1. (¬P → Q)	A
2	2. ¬Q	A
1,2	3. ¬¬P	1,2 MTT
1,2	4. P	3 DN

Homework #3: p. 18, 1a,b,c,d,e

Work in groups. I will collect only one paper from each group. You may assign a single scribe for the group, or divide up the writing.

Problems from assignment #2.

A. Show that $((P \,\&\, Q) \rightarrow \neg\neg R)$ is a well formed formula.

1. P (rule 1)
2. Q (rule 1)
3. (P & Q) (rule 3, applied to 1,2)
4. R (rule 1)
5. ¬R (rule 2, applied to 4)
6. ¬¬R (rule 2, applied to 5)
7. ((P & Q) → ¬¬R) (rule 3, applied to 3,5)

Different derivations may be obtained by changing order, or by adding useless steps. For example, we may insert 3.5. ¬Q, or we may switch steps 3 and 4.

B. Let A stand for "Alice minds the baby", B--"The baby cries," C--"The Cheshire cat makes trouble". Then we have the following translations (there are other possibilities).

1. (A & (B & C))
Alice is minding the baby, and the baby is crying while the Cheshire cat makes trouble.
2. ((A & B) & C)
Alice is minding the baby, who is crying, and the Cheshire cat is making trouble.
3. (¬(A & B) & C)
It is not the case that Alice minds the baby while the baby cries, but the Cheshire cat is making trouble. We don't have Alice minding the crying baby, but the Cheshire cat is making trouble anyway.
4. (A & ¬(B & C))
Alice is minding the baby, and we do not have the baby crying in addition to the Cheshire cat's making trouble.
5. (A & (B ∨ C))
Alice is minding the baby, and either the baby cries or the Cheshire cat makes trouble.
6. ((A → B) & C)
The baby cries if Alice is minding it, and the Cheshire cat makes trouble.
7. (¬(A → B) & C)
It is not necessarily so that the baby cries if Alice minds it, but the Cheshire cat makes trouble regardless.
8. (A → ¬(B & C))
If Alice minds the baby, then at least we won't have both the baby crying and the Cheshire cat making trouble.
9. (A → (B∨ C))
If Alice minds the baby, then either the baby cries or else the Cheshire cat makes trouble.
10. ((A → B) ∨ C)
If Alice minds the baby, it cries, unless the Cheshire cat is making trouble.

C. Premises:
1. If the river floods, then the wheat crop will be destroyed.
2. The river will flood if there is an early thaw.
3. In any case, there will be heavy rains later in the summer.

Conclusion: If there is an early thaw, then our community will be bankrupt unless there are heavy rains later in the summer.

Letting P stand for "the river floods", Q--"the wheat crop is destroyed", R--"there is an early thaw", S--"there are heavy rains later in the summer", T--" the community will be bankrupt", the argument above has this form.

Premises: 1. (P → Q)

2. (R → P)
3. S

Conclusion: (R→ (¬S → T)) (not (R → ¬(S → T))

Lecture 4

Collect homework, one paper for each group.

Spend a little time on Assignment #2, possibly some on Assignment #3. Then describe next rule of inference.

How did the arguments go ?

p. 18: 1a,b,c,d,e

(a) (P → (P → Q)), P |- Q

1	1. P		A
2	2. (P → (P → Q))		A
1,2	3. (P → Q)	2,1	MPP
1,2	4. Q	3,1	MPP

(b) (Q → (P → R)), ¬R, Q |- ¬P

1	1. Q		A
2	2. (Q → (P → R))		A
1,2	3. (P → R)	2,1	MPP
4	4. ¬R	A	
1,2,4	5. ¬P 3,4	MTT	

(c) (P → ¬¬Q), P |- Q

1	1. P		A
2	2. (P → ¬¬Q)	A	
1,2	3. ¬¬Q	2,1	MPP
1,2	4. Q	3	DN

(d) (¬¬Q → P), ¬P |- ¬Q

1	1. (¬¬Q → P)	A	
2	2. ¬P	A	
1,2	3. ¬¬¬Q	1,2	MTT
1,2	4. ¬Q 3	DN	

(e) (¬P → ¬Q), Q |- P

1	1. (¬P → ¬Q)		A	
2	2. Q			A
1,2	3. ¬¬Q		2	DN
1,2	4. ¬¬P	1,3	MTT	
1,2	5. P		4	DN

So far, we have four rules of inference: A, MPP, MTT, and DN (really two rules in one). There are further rules.

The Next one is

5. <u>Conditional Proof</u> (CP)

This rule is used in proving statements of the form (F → G), conditional statements. The natural way to argue for this statement is to suppose F temporarily, and try to to prove G. Having succeeded, you conclude that (F → G), no longer assuming F. The rule makes this legitimate.

It is important to pay attention to the decorations, especially the ones on the left, standing for the assumptions used. Pattern is different.

When we write F, it is an assumption (temporarily). When we arrive at G, F is likely to be one of the assumptions listed on the left. But, when we write (F → G), F is no longer one of the assumptions.

For other rules, MPP, MTT, the list of assumptions grew.

We write something like the following:

7		7. F		A
.		.		.
.		.		.
.		.		.
4,7,12	13. G		.	
4,12		14. (F → G)	7,13	CP

Note that 7 appears on the left in line 13, not in line 14.

<u>Example 1</u>: (P → Q) |- (¬Q → ¬P)

1	1. (P → Q)			A
2	2. ¬Q		A (temporary assumption, not one allowed at the end)	
1,2	3. ¬P	1,2	MTT	
1	4. (¬Q → ¬P)	2,3	CP	

If we do not get rid of the temporary assumption, we have not done what we set out to do.

Example 2: |- ((P → Q) → ((Q → R) → (P → R)))

1	1.	(P → Q)		A
2	2.	(Q → R)		A
3	3.	P		A
1,3	4.	Q	3,1	MPP
1,2,3	5.	R	4,2	MPP
1,2	6.	(P → R)	3,5	CP
1	7.	((Q → R) → (P → R))	2,6	CP
	8.	((P → Q) → ((Q → R) → (P → R)))	1,7	CP

Here we got rid of all three temporary assumptions.

The next examples are very short.

Example 3: |- (P → P)

1	1.	P		A
	2.	(P → P)	1,1	CP

This example has the odd feature that the lines to which we apply the CP rule are the same.

Example 4: P |- (Q → P)

1	1.	Q		A
2	2.	P		A
2	3.	(Q → P)	1,2	CP

Problems from assignment #4

A. p. 18: 1f,g,h,i,j,k,l,m,n

(f) (P → ¬Q) |- (Q → ¬P)

1	1.	(P → ¬Q)		A
2	2.	Q		A
2	3.	¬¬Q	2	DN
1,2	4.	¬P	1,3	MTT
1	5.	(Q → ¬P)	2,4	CP

(g) (¬P → Q) |- (¬Q → P)

1	1.	(¬P → Q)		A
2	2.	¬Q		A
1,2	3.	¬¬P	1,2	MTT
1,2	4.	P	3	DN
1	5.	(¬Q → P)	2,4	CP

(h) (¬P → ¬Q) |- (Q → P)

1	1.	(¬P → ¬Q)		A
2	2.	Q		A
2	3.	¬¬Q	2	DN
1,2	4.	¬¬P	1,3	MTT
1,2	5.	P	4	DN
1	6.	(Q → P)	2,5	CP

(i) (P → Q), (Q → R) |- (P → R)

1	1.	(P → Q)		A
2	2.	(Q → R)		A
3	3.	P		A
1,3	4.	Q	3,1	MPP
1,2,3	5.	R	4,2	MPP
1,2	6.	(P → R)	3,5	CP

(j) (P → (Q → R)) |- ((P → Q) → (P → R))

1	1.	(P → (Q → R))		A
2	2.	(P → Q)		A
3	3.	P		A
1,3	4.	(Q → R)	3,1	MPP
2,3	5.	Q	3,2	MPP
1,2,3	6.	R	5,4	MPP
1,2	7.	(P → R)	3,6	CP
1	8.	((P → Q) → (P → R))	2,7	CP

(k) (P → (Q → (R → S))) |- (R → (P → (Q → S)))

1	1.	(P → (Q → (R → S)))		A
2	2.	R		A
3	3.	P		A
4	4.	Q		A
1,3	5.	(Q → (R → S))	3,1	MPP
1,3,4	6.	(R → S)	4,5	MPP
1,2,3,4	7.	S	2,6	MPP
1,2,3	8.	(Q → S)	4,7	CP
1,2	9.	(P → (Q → S))	3,8	CP
1	10.	(R → (P → (Q → S)))	2,9	CP

(l) (P → Q) |- ((Q → R) → (P → R))

1	1.	(P → Q)		A
2	2.	(Q → R)		A
3	3.	P		A
1,3	4.	Q	3,1	MPP
1,2,3	5.	R	4,2	MPP
1,2	6.	(P → R)	3,5	CP
1	7.	((Q → R) → (P → R))	2,6	CP

(m) P |- ((P → Q) → Q)

1	1.	P		A
2	2.	(P →Q)		A
1,2	3.	Q	1,2	MPP
1	4.	((P → Q) → Q)	2,3	CP

(n) P |- ((¬(Q → R) → ¬P) → (¬R → ¬Q))

1	1.	P		A
2	2.	(¬(Q → R) → ¬P)		A
3	3.	¬R		A
1	4.	¬¬P	1	DN
1,2	5.	¬¬(Q → R)	2,4	MTT
1,2	6.	(Q → R)	5	DN
1,2,3	7.	¬Q	6,3	MTT
1,2	8.	(¬R → ¬Q)	3,7	CP
1	9.	((¬(Q → R) → ¬P) → (¬R → ¬Q))	2,8	CP

B. For statements of the following form give the best translation, substituting a single logical connective for the word or words underlined.

1. P but Q (P & Q)
2. Q unless P (P ∨ Q)
3. P implies Q (P → Q)
4. P only if Q (P → Q)
5. P while Q (P & Q)
6. P precisely when Q (P ↔ Q)
7. it is not the case that P ¬P

Problems from Assignment #3

p. 18: 1a,b,c,d,e

(a) (P → (P → Q)), P |- Q

1	1. P			A
2	2. (P → (P → Q))			A
1,2	3. (P → Q)		2,1	MPP
1,2	4. Q		3,1	MPP

(b) (Q → (P → R)), ¬R, Q |- ¬P

1	1. Q			A
2	2. (Q → (P → R))			A
1,2	3. (P → R)		2,1	MPP
4	4. ¬R		A	
1,2,4	5. ¬P	3,4	MTT	

(c) (P → ¬¬Q), P |- Q

1	1. P			A
2	2. (P → ¬¬Q)		A	
1,2	3. ¬¬Q		2,1	MPP
1,2	4. Q		3	DN

(d) (¬¬Q → P), ¬P |- ¬Q

1	1. (¬¬Q → P)		A	
2	2. ¬P		A	
1,2	3. ¬¬¬Q		1,2	MTT
1,2	4. ¬Q	3	DN	

(e) (¬P → ¬Q), Q |- P

1	1. (¬P → ¬Q)		A	
2	2. Q			A
1,2	3. ¬¬Q		2	DN
1,2	4. ¬¬P	1,3	MTT	
1,2	5. P		4	DN

Lecture 5

Return homework on first 4 rules, collect homework on CP rule, more translation

Swap group membership so that no group has size > 4.

We have five rules of inference so far: A, MPP, MTT, DN, CP. More to come.

Sixth Rule: <u>& Introduction</u> (&I). This rule is used in proving statements of the form (F & G).

Given F and G, we may conclude (F & G). The assumptions used are those behind F, together with those behind G, so these are the numbers on the left. On the right, the numbers are the lines of F and G.

<u>Example</u>: P, (P → Q), (P → R) |- (Q & R)

1	1. P		A
2	2. (P → Q)		A
3	3. (P → R)		A
1,2	4. Q	1,2	MPP
1,3	5. R	1,3	MPP
1,2,3	6. (Q&R)	4,5	&I

Seventh Rule: <u>& Elimination</u> (&E). As the name suggests, this rule is the opposite of & Introduction. Like DN, it is two rules in one.

(a) From (F & G), we may conclude F (b) From (F & G), we may conclude G

The assumptions are those behind (F&G), so these are the numbers on the left. On the right, the number is that of (F & G).

<u>Example</u>: (P → R), (S → ¬Q) |- ((P & Q) → (R & ¬S))

1	1. (P → R)			A
2	2. (P & Q)			A
2	3. P		2	&E
1,3	4. R		3,1	MPP
5	5. (S → ¬Q)			A
2	6. Q		2	&E
2	7. ¬¬Q		6	DN
5,2	8. ¬S	5,7	MTT	
1,2,5	9. (R & ¬S)		4,8	&I
1,5	10. ((P & Q) → (R & ¬S))		2,9	CP

Lecture 6

Return homework #4, with solutions.

Questions ?

So far, we have 7 rules: A, MPP, MTT, DN, CP, &E (new last time), &I (also new last time). Still more rules.

Eighth Rule: <u>∨ Introduction</u> (∨I)

This rule is two in one.

(a) From F, we may conclude (F ∨ G) (b) From G, we may conclude (F ∨ G).

The assumptions are those behind F for (a), or G for (b), so these are the numbers on the left. On the right, we list the line of F for (a), or G for (b).

<u>Example</u>: (P & Q) |- (P ∨ Q)

1	1. (P & Q)		A
1	2. P	1	&E
1	3. (P ∨ Q)	2	∨ E

Ninth Rule: <u>∨ Elimination</u> (∨E)

This rule says that if we can prove H from F and we can also prove H from G, then we may conclude H from (F ∨ G). So, we list H three times.

The decoration is a little complicated. The assumptions listed on the left are are those behind (F ∨ G), the assumptions other than F used with F to get H, and the assumptions other than G used with G to get H. On the right, we list the line of (F ∨ G), the line of F, the line where we obtained H using F, the line of G, and the line where we obtained H from G.

The rule is natural. Say you want to convince a friend that it is always possible to get to Chicago by train, arriving before noon. When Chicago is on daylight savings time, there is no time difference. There is an 8:00 train, arriving at about 11:00. When Chicago is on standard time, South Bend is an hour ahead of Chicago, so you could leave as late as 9:00 and still arrive at about 11:00. Your friend is going to believe that Chicago is on daylight savings time or not.

<u>Example 1</u>: ((P & Q) ∨ (R & P)) |- P

1	1. (P & Q) ∨ (R & P)		A	
2	2. (P & Q)			A
2	3. P	2		&E

4	4. (R & P)		A
4	5. P	4	&E
1	6. P	1,2,3,4,5	∨E

Example 2: (P & (Q ∨ R)) |- ((P & Q) ∨ (P & R))

1	1. (P & (Q ∨ R))		A
1	2. P	1	&E
1	3. (Q ∨ R)	1	&E
4	4. Q		A
1,4	5. (P & Q)	2,4	&I
1,4	6. ((P & Q) ∨ (P & R))	5	∨I
7	7. R		A
1,7	8. (P & R)	2,7	&I
1,7	9. ((P & Q) ∨ (P & R))	8	∨I
1	10. ((P & Q) ∨ (P & R))	3,4,6,7,9	∨E

Tenth Rule: Reductio ad absurdum (RAA)

This rule says that if assuming F, we can prove a statement of the form (G & ¬G), then we may conclude ¬F, dropping the assumption F. For decorations, on the right, we refer to the line where we supposed F and the line where we obtained (G & ¬G). On the left, we give the assumptions, except possibly for F, behind (G & ¬G).

Example 1: P,¬R |- ¬(P → R)

1	1. P	A	
2	2. ¬R A		
3	3. (P → R)	A	
1,3	4. R	1,3	MPP
1,2,3	5. (R & ¬R)	4,2	&I
1,2	6. ¬(P → R)	3,5	RAA

Homework #4: p. 27:1a,b,c,d,e,f

Lecture 7

Collect homework, call for questions, volunteers.

Last time, I introduced rule RAA, gave just one example of use.

Example 2: ¬P ⊢ (P → R)

1	1. ¬P A		
2	2. P	A	
3	3.¬R	A	
1,2	4. (P&¬P)	1,2	&I
1,2	5. ¬¬R	3,4	RAA
1,2	6. R	5	DN
1	7. (P → R)	2,6	CP

Example 3: ⊢ (P ∨ ¬P)

1	1. ¬(P ∨ ¬P)		A	
2	2. P			A
2	3. (P ∨ ¬P)		2	∨I
1,2	4. ((P ∨ ¬P) & ¬(P ∨ ¬P))		3,1	&I
1	5. ¬P	2,4	RAA	
1	6. (P ∨ ¬P)		5	∨I
1	7. ((P ∨ ¬P) & ¬(P ∨ ¬P))		6,1	&I
	9. ¬¬(P ∨ ¬P)	1,7	RAA	
	10. (P ∨ ¬P)		9	DN

Homework #5: p. 27: 1 g,h,i,j

Solutions to homework problems #4, #5

p. 27: 1a,b,c,d,e,f,g,h,i,j

(a) P |- (Q → (P&Q))

1	1. P		A
2	2. Q		A
1,2	3. (P & Q)	1,2	&I
1	4. (Q → (P & Q))	2,3	CP

(b) (P & (Q & R)) |- (Q & (P & R))

1	1. (P & (Q & R))		A
1	2. P	1	&E
1	3. (Q & R)	1	&E
1	4. Q	3	&E
1	5. R	3	&E
1	6. (P & R)	2,5	&I
1	7. (Q & (P & R))	4,6	&I

(c) ((P → Q) & (P → R)) |- (P → (Q & R))

1	1. ((P → Q) & (P → R))		A
2	2. P		A
1	3. (P → Q)	1	&E
1	4. (P → R)	1	&E
1,2	5. Q	2,3	MPP
1,2	6. R	2,4	MPP
1,2	7. (Q & R)	5,6	&I
1	8. (P → (Q & R))	2,7	CP

(d) Q |- (P ∨ Q)

1	1. Q		A
1	2. (P ∨ Q)	1	∨I

(e) (P & Q) |- (P ∨ Q)

1	1. (P & Q)		A
1	2. P	1	&E
1	3. (P ∨ Q)	2	∨I

(f) ((P → Q) & (Q → R)) |- ((P ∨ Q) → R)

1	1. ((P → Q) & (Q → R))		A
1	2. (P → R)	1	&E
1	3. (Q → R)	1	&E
4	4. (P ∨ Q)		A
5	5. P		A
1,5	6. R	5,2	MPP
7	7. Q		A
1,7	8. R	7,3	MPP
1,4	9. R	4,5,6,7,8	∨E
1	10. ((P ∨ Q) → R)	4,9	CP

(g) (P → Q),(R → S) |- ((P & R) → (Q & S))

1	1. (P → Q)		A
2	2. (R → S)		A
3	3. (P & R)		A
3	4. P	3	&E
3,1	5. Q	4,1	MPP
3	6. R	3	&E
3,2	7. S	5,2	MPP

1,2,3	8. (Q & S)	5,7	&I
1,2	9. ((P & R) → (Q & S))	3,8	CP

(h) (P → Q), (R → S) |- ((P ∨ R) → (Q ∨ S))

1	1. (P → Q)		A
2	2. (R → S)		A
3	3. (P ∨ R)		A
4	4. P		A
1,4	5. Q	4,1	MPP
1,4	6. (Q ∨ S)	5	∨I
7	7. R		A
2,7	8. S	7,2	MPP
2,7	9. (Q ∨ S)	8	∨I
1,2,3	10. (Q ∨ S)	3,4,6,7,9	∨E
1,2	11. ((P ∨ R) → (Q ∨ S))	3,10	CP

(i) (P → (Q & R)) |- ((P → Q) & (P → R))

1	1. (P → (Q & R))		A
2	2. P		A
1,2	3. (Q & R)	2,1	MPP
1,2	4. Q	3	&E
1	5. (P → Q)	2,4	CP
1,2	6. R	3	&E
1	7. (P → R)	2,6	CP
1	8. ((P → Q) & (P → R))	5,7	&I

(j) ¬P → P |- P

1	1. ¬P → P		A
2	2. ¬P	A	
1,2	3. P	1,2	MPP
1,2	4. (P & ¬P)	2,3	&I
1	5. ¬¬P2,4	RAA	
1	6. P	5	DN

Lecture 8

Collect homework and hand out solutions.

We have 10 rules, so far. There are no rules for ↔. What the author does is say that this is not a basic symbol after all. We will go ahead and use it wherever we please. But, we say that (F ↔ G) is an abbreviation for ((F → G) & (G→F)).

Example: P, (P → Q) |- (P ↔ Q) = ((P → Q) & (Q → P))

1	1. (P→Q)		A
2	2. Q		A
3	3. P		A
3	4. (Q→P)	2,3	CP
1,3	5. ((P→Q) & (Q→P))	1,4	&I

We would like our set of rules to have the following feature:

Soundness: If $F_1,...,F_n$ |- G, and $F_1,...,F_n$ are true, then G must be true.

This boils down to showing that for each single step in a proof, applying a single rule, if the assumptions behind the statement (the ones listed on the left) are all true, then so is the statement itself.

When we introduced first few rules of inference, I asked you to think about soundness. Daily had a problem with MTT.

MTT says: from (F → G), ¬G, conclude ¬F.
If (F → G) is true and ¬G is true, then F can't be true, for then both G and ¬G would be true. This doesn't happen, so, F can't be true. Must ¬F be true ? Daily expressed doubts about this, and he is not alone.

Let's take a closer look at some of the other rules.

What about DN ?
DN says:
(a) From F, conclude ¬¬F, and
(b) from ¬¬F, conclude F.
(a) If F is true, must ¬¬F be true ? Virtually everyone accepts this.
(b) If ¬¬F is true, must F be true ? The people who have reservations about MTT generally also have reservations about this half of DN.

What about RAA. ?
RAA says: If, temporarily assuming F, you can conclude (G&¬G), then you may conclude ¬F, dropping the assumption F.

If, whenever F is true, (G & ¬G) is true, then F cannot be true. Must ¬F be true ? Again some people have reservations.

The other rules (A, MPP, CP, &I, &E, ∨I, ∨E) are all pretty generally accepted.

Woods asked whether there were really different "logics". In "classical" logic, the rules may vary slightly, but the same things are provable. There are "non-classical" systems, some probably more to Daily's liking, which do not prove the same things.

I said on the first day that I will have you write papers. You might decide to describe an alternate system. I'll say more about the papers right after the first exam.

To justify using classical rules, we adopt a classical view of <u>truth</u>, in which ¬F is true just in case F is not true. I'll say more about truth later, but just this much is enough to take care of the problems with Modus Tollens, DN, and RAA.

If we believe that all of our rules are sound, in the sense that the conclusions are true provided all of the (undischarged) assumptions are true, then we could argue for Soundness.

Let us assume Soundness, for now.

How could we show it is not the case that $F_1,...,F_n$ |- G ?

Assuming Soundness, it is enough to give an interpretation; i.e., to assign meanings to the propositional variables, in such a way that $F_1,...,F_n$ are all true and G is false.

<u>Example 1</u>: It is not the case that (P → Q) |- (Q → P).

We show this by giving an appropriate interpretation.

<u>First interpretation</u>: Let P--there have been heavy rains, Q--the river rises. Under this interpretation, (P → Q) says that if there have been heavy rains, then the river rises, which is true. The other statement, (Q→P) says that if the river rises, then there have been heavy rains. However, the river rises when it warms up after a long spell of snow and cold, even without rain.

<u>Second interpretation</u>: Let A--x is even, B--y is even, P--(A & B), Q--x+y is even. Under this interpretation, (P → Q) says that if x and y are even, then the sum is even, which is true. The other statement, (Q → P) says that if x+y is even, then x and y are both even. This is false, since x+y is also even if x and y are both odd.

In addition to Soundness, we would like our set of rules to have another important feature.

<u>Completeness</u>: If G is true whenever $F_1,...,F_n$ are all true, then $F_1,...,F_n$ |- G.

Woods asked whether we really <u>could</u> prove everything--everything that we would want to prove. The question is whether our system is complete. In fact, it is, but understanding why depends on precise notion of truth.

Solutions to homework problems

A. p. 27: 2.

(a) $P \nvdash (P \& Q)$

P--It is summer

Q--It is winter

The right-hand side says that it is simultaneously summer and winter, which is never true. The left-hand side is true in summer.

(b) $(P \vee Q) \nvdash P$

P--It is summer

Q--It is winter

The left-hand side is true in winter, but the right-hand side is not.

(c) $(P \vee Q) \nvdash (P \& Q)$

P--It is summer

Q--It is winter

The right-hand side is never true. The left-hand side is true in summer or in winter.

(d) $(P \rightarrow Q) \nvdash (P \& Q)$

P--It is summer

Q--birds are singing by 6:00 a.m.

The left-hand side says that if it is summer, the birds are singing by 6:00 a.m., true where I live, near campus. The right-hand side says that it <u>is</u> summer, and the birds...etc. This is not true right now.

Solutions to #6, problems on pp. 33, 40

p. 33: 1a,b

(a) Q, (P ↔ Q) |- P

1	1. Q		A
2	2. ((P → Q) & (Q → P))		A
2	3. (Q → P)	2	&E
1,2	4. P	1,3	MPP

(b) (P → Q), (Q → P) |- (P ↔ Q)

1	1. (P → Q)		A
2	2. (Q → P)		A
1,2	3. (P → Q) & (Q → P)	1,2	&I

C. p. 40: 1a,b,c,d.

(a) (P ∨ Q) |- (P ∨ Q)

1	1. (P ∨ Q)	A

(b) (P & P) |- P and P |- (P & P) (two separate proofs to show interderivability)

1	1. (P & P)		A
1	2. P	1	&E

1	1. P		A
1	2. (P&P)	1	&I

(c) (P & (Q ∨ R)) |- ((P & Q) ∨ (P & R)) and ((P & Q) ∨ (P & R)) |- (P & (Q ∨ R))

1	1. (P & (Q ∨ R))		A
1	2. P	1	&E
1	3. (Q ∨ R)	1	&E
4	4. Q		A
1,4	5. (P & Q)	2,4	&I
1,4	6. ((P & Q) ∨ (P & R))	5	∨I
7	7. R		A
1,7	8. (P & R)	2,7	&I
1,7	9. ((P & Q) ∨ (P & R))	8	∨I
1	10. ((P & Q) ∨ (P & R))	3,4,6,7,9	∨E

1	1. ((P & Q) ∨ (P & R))		A
2	2. (P & Q)		A
2	3. P	2	&E
2	4. Q	2	&E
2	5. (Q ∨ R)	4	∨I
2	6. (P & (Q ∨ R))	3,5	&I

7	7. (P & R)		A
7	8. P	7	&E
7	9. R	7	&E
7	10. (Q ∨ R)	9	∨I
7	11. (P & (Q ∨ R))	8,10	&I
1	12. (P & (Q ∨ R))	1,2,6,7,11	∨E

Lecture 9

Collect homework and hand out solutions.

Remember Exam. I on Friday, in class.
Help session Thursday evening.

What sorts of things will be on exam.? If you can do all of the homework problems, you will do well on the exam.

A. Translation.

B. Proofs (this is the biggest item). Will ask you to give whole proofs, as on homework. May also give some partial proofs, with decorations to be filled in.

C. Definition of well-formed formula.

Recall how the definition goes.
1. A propositional variable is a well formed formula.
2. If F is a well formed formula, then so is ¬F.
3. If F and G are well formed formulas, then so is (F_G), where _ is &, ∨, or → (I had said ↔, but we have now dropped that from list of basic connectives).
4. Nothing is a formula unless it can be obtained by finitely many steps using 1, 2, 3.

<u>Example</u>: Show that ¬(P → Q) is a well-formed formula, by giving a finite sequence of steps using 1, 2, and 3.

1. P
2. Q
3. (P → Q)
4. ¬(P → Q)

<u>Example</u>: Show that P¬ is not a well-formed formula.

If this is a formula, it is obtained by a finite sequence of steps of types 1, 2, 3. Consider the last step.

Not type 1, not single propositional variable.
Not type 2, doesn't start with ¬.
Not type 3, doesn't start with (.

D. Show that there is no proof of statement G from some assumptions $F_1,...,F_n$ (as on today's homework). Give interpretation showing that $F_1,...,F_n$ may all be true while G is false.

Sample proofs using ∨E

(1) (P ∨ Q), (P → R), (Q → R) |- R

1	1. (P ∨ Q)		A	
2	2. (P → R)		A	
3	3. (Q → R)		A	
4	4. P		A	
2,4	5. R	4,2	MPP	
6	6. Q		A	
3,6	7. R	6,3	MPP	
2,3	8. R	1,4,5,6,7		∨E

(2) (P ∨ Q), ¬P |- Q

1	1. (P ∨ Q)	A		
2	2. ¬P A			
3	3. P	A		
4	4. ¬Q A			
2,3	5. (P & ¬P)	3,2 &I		
2,3	6. ¬¬Q	4,5	RAA	
2,3	7. Q	6	DN	
8	8. Q	A		
1,2	9. Q	1,3,7,8,8		∨E

(3) (¬P ∨ Q) |- (P → Q)

1	1. (¬P ∨ Q)	A		
2	2. P	A (aim for Q and apply CPP)		
3	3. ¬P A (use ∨E)			
4	4. ¬Q A			
2,3	5. (P & ¬P)	2,3	&I	
2,3	6. ¬¬Q	4,5	RAA	
2,3	7. Q	6	DN	
8	8. Q	A		
1,2	9. Q	1,3,7,8,8		∨E
1	10. (P → Q)	2,9	CP	

Homework #7: p. 41: 1d,e,f,g,h,i,j

(d) (P ∨ (Q & R)) |- ((P ∨ Q) & (P ∨ R)) and ((P ∨ Q) & (P ∨ R)) |- (P ∨ (Q & R))

1	1. (P ∨ (Q & R))		A

Prove ((P ∨ Q) & (P ∨ R)) first assuming P, then assuming (Q & R), combine by ∨E.

2	2. P		A
2	3. (P ∨ Q)	2	∨I
2	4. (P ∨ R)	2	∨I
2	5. ((P ∨ Q) & (P ∨ R))	3,4	&I
6	6. (Q & R)		A
6	7. Q	6	&E
6	8. R	6	&E
6	9. (P∨Q)	7	∨I
6	10. (P∨R)	8	∨I
6	11. ((P ∨ Q) & (P ∨ R))	9,10	&I
1	12. ((P ∨ Q) & (P ∨ R))	1,2,5,6,11	∨E

1	1. ((P ∨ Q) & (P ∨ R))		A
1	2. (P ∨ Q)	1	&E
1	3. (P ∨ R)	1	&E

Prove (P ∨ (Q & R)) first assuming P, then assuming Q (and using (P∨R)). Combine by ∨E.

4	4. P		A
4	5. (P ∨ (Q & R))	4	∨I
6	6. Q		A

To use (P ∨ R) in proving ((P ∨ (Q & R)), assuming Q, we have a proof from P, so we aim for one from R, and combine by ∨E.

7	7. R		A
6,7	8. (Q & R)	6,7	&I
6,7	9. (P ∨ (Q & R))	8	∨I
1,3,6	10. (P ∨ (Q & R))	3,4,5,7,9	∨E
1	11. (P ∨ (Q & R))	2,4,5,6,10	∨E

(e) (P & Q) -||- ¬(P → ¬Q)

1	1. (P & Q)	A	
1	2. P	1 &E	
1	3. Q	1 &E	
4	4. (P → ¬Q)	A (aim for contradiction and use RAA to get ¬(P → ¬Q))	
1,4	5. ¬Q	2,4 MPP	
1,4	6. (Q & ¬Q)	3,5 &I	
1	7. ¬(P → ¬Q)	4,6 RAA	

1	1. ¬(P→¬Q)	A	

2	2. ¬P	A (aim for contradiction and use RAA to get ¬¬P)
3	3. P	A (aim for ¬Q and use CP to get (P→¬Q))
4	4. Q	A (aim for contradiction and use RAA to get ¬Q)
2,3	5. P&¬P	3,2 &I
2,3	6. ¬Q	4,5 RAA
2	7. P→¬Q	3,6 CP
1,2	8. (P→¬Q)&¬(P→¬Q)	7,1 &I
1	9. ¬¬P	2,8 RAA
1	10. P	9 DN
11	11. ¬Q	A (aim for contradiction and use RAA to get ¬¬Q)
12	12. P	A (use CP to get (P→¬Q))
11	13. (P → ¬Q)	12,11 CP
1,11	14. (P→¬Q)&¬(P→¬Q)	13,1 &I
1	15. ¬¬Q	11,14 RAA
1	16. Q	15 DN
1	17. P&Q	10,16 &I

(f) ¬(P ∨ Q) -||- (¬P & ¬Q)

1	1. ¬(P ∨ Q)	A
2	2. P	A (aim for contradiction and use RAA to get ¬P)
2	3. (P ∨ Q)	2 ∨I
1,2	4. ((P ∨ Q) & ¬(P ∨ Q))	1,3 &I
1	5. ¬P	2,4 RAA
6	6. Q	A (aim for contradiction and use RAA to get ¬Q)
6	7. (P ∨ Q)	6 ∨I
1,6	8. ((P ∨ Q) & ¬(P ∨ Q))	7,1 &I
1	9. ¬Q	6,8 RAA
1	10. (¬P & ¬Q)	5,9 &I

1	1. (¬P & ¬Q)		A
2	2. (P ∨ Q)		A

(aim for contradiction from 1,2 and use RAA; prove (Q&¬Q) from P and then from Q)

3	3. P		A
4	4. ¬(Q & ¬Q)		A
1	5. ¬P	1	&E
1,3	6. (P & ¬P)	3,5	&I
1,3	7. ¬¬(Q & ¬Q)	4,6	RAA
1,3	8. (Q & ¬Q)	7	DN
9	9. Q		A
1	10. ¬Q	1	&E
1,9	11. (Q & ¬Q)	9,10	&I
1,2	12. (Q & ¬Q)	2,3,8,9,11	∨E
1	13. ¬(P ∨ Q)	2,12	RAA

(g) ¬(P & Q) -||- (¬P ∨ ¬Q)

1 1. ¬(P & Q) A
2 2. ¬(¬P ∨ ¬Q) A (aim at contradiction and use RAA, DN to get (¬P∨¬Q))
3 3. ¬P A (aim at contradiction and use RAA, DN to get P)
3 4. (¬P ∨ ¬Q) 3 ∨I
2,3 5. ((¬P∨¬Q) & ¬(¬P∨¬Q)) 4,2 &I
2 6. ¬¬P 3,5 RAA
2 7. P 6 DN
8 8. ¬Q A (aim at contradiction and use RAA, DN to get Q)
8 9. (¬P∨¬Q) 8 ∨I
2,8 10. ((¬P∨¬Q)&¬(¬P∨¬Q)) 9,2 &I
2 11. ¬¬Q 8,10 RAA
2 12. Q 11 DN
2 13. (P & Q) 7,12 &I
1,2 14. ((P & Q) & ¬(P & Q)) 13,1 &I
1 15. ¬¬(¬P ∨ ¬Q) 2,14 RAA
1 16. (¬P ∨ ¬Q) 15 DN

1 1. (¬P ∨ ¬Q) A (use ∨E, proving ¬(P & Q) first from ¬P and then from ¬Q)
2 2. ¬P A
3 3. (P & Q) A (aim at contradiction and use RAA to get ¬(P & Q)
3 4. P 3 &E
2,3 5. (P & ¬P) 4,2 &I
2 6. ¬(P & Q) 3,5 RAA
7 7. ¬Q A
8 8. (P & Q) A (as above, aim at contradiction and use RAA to get ¬(P & Q))
8 9. Q 8 &E
7,8 10. (Q & ¬Q) 9,7 &I
7 11. ¬(P & Q) 8,10 RAA
1 12. ¬(P & Q) 1,2,6,7,11 ∨E

(h) (P & Q) -||- ¬(¬P ∨ ¬Q)

1 1. (P&Q) A
2 2. (¬P ∨ ¬Q) A (aim at contradiction and use RAA;
prove ¬(P & Q) first from ¬P and then from ¬Q, and then use ⁄Æ)
3 3. ¬P A
1 4. P 1 &E
1,3 5. (P & ¬P) 4,3 &I
3 6. ¬(P & Q) 1,5 RAA
7 7. ¬Q A
1 8. Q 1 &E
1,7 9. (Q & ¬Q) ,7 &I
7 10. ¬(P & Q) 1,9 RAA

2	11.	¬(P & Q)	2,3,6,7,10 ∨E
1,2	12.	((P & Q) & ¬(P & Q))	&I
1	13.	¬(¬P ∨ ¬Q)	2,12 RAA

1	1.	¬(¬P ∨ ¬Q)	A
2	2.	¬P	A (aim at contradiction and use RAA, DN to get P)
2	3.	(¬P ∨ ¬Q)	2 ∨I
1,2	4.	(¬P ∨ ¬Q) & ¬(¬P ∨ ¬Q)	3,1 &I
1	5.	¬¬P	2,4 RAA
1	6.	P	5 DN
7	7.	¬Q	A (aim at contradiction and use RAA, DN to get Q)
7	8.	(¬P ∨ ¬Q) 7	∨I
1,7	9.	(¬P ∨ ¬Q) & ¬(¬P ∨ ¬Q)	8,1 &I
1	10.	¬¬Q	7,9 RAA
1	11.	Q	10 DN
1	12.	(P & Q)	6,11 &I

(i) (P → Q) |- (¬P ∨ Q)

1	1.	P → Q	A
2	2.	¬(¬P ∨ Q)	A (aim for contradiction and use RAA, DN to get ¬P⁄Q)
3	3.	P	A (aim for contradiction and use RAA to get ¬P)
1,3	4.	Q	3,1 MPP
1,3	5.	(¬P ∨ Q)	4 ∨I
1,2,3	6.	(¬P∨Q) & ¬(¬P∨Q)	5,2 &I
1,2	7.	¬P	3,6 RAA
1,2	8.	¬P∨Q	7 ⁄I
1,2	9.	(¬P∨Q) & ¬(¬P∨Q)	8,2 &I
1	10.	¬¬(¬P∨Q)	2,9 RAA
1	11.	¬P∨Q	10 DN

(j) (¬P → Q) |- (P ∨ Q)

1	1.	(¬P → Q)	A
2	2.	¬(P ∨ Q)	A (aim for contradiction use RAA, DN to get P⁄Q)
3	3.	P	A (aim for contradiction and use RAA to get ¬P)
3	4.	(P ∨ Q)	3 ∨I
2,3	5.	((P ∨ Q) & ¬(P ∨ Q))	4,2 &I
2	6.	¬P	3,5 RAA
1,2	7.	Q	6,1 MPP
1,2	8.	(P ∨ Q)	7 ∨I
1,2	9.	((P ∨ Q) & ¬(P ∨ Q))	8,9 &I
1	10.	¬¬(P ∨ Q)	2,9 RAA
1	11.	(P ∨ Q)	10 DN

p. 62: 1a,b

Lecture 10

Collect homework and hand out solutions to remaining problems on p.41 (very hard).

Announce help session Thursday, 7-9 p.m. MCC, Room 214.

I hope you found the problems on p. 41 challenging. I did. You should feel good if you got some of the problems on p. 41, or parts of problems. Why assign such difficult ones ? Some people enjoy a challenge. In addition, I ask you to consider what a machine could do. Certainly, a machine could check proofs. Could a machine think up proofs ? We will come back to this point.

Harder problems illustrate the need for strategy. You do not want to introduce an extra assumption unless you have a plan for discharging it. Let us look at this in connection with some problems. First, the ones on p. 61.

(a) |- (Q → R) → ((P → Q) → (P → R))

1	1. (Q → R)	A (aim for ((P → Q) → (P → R)) and use CP)
2	2. (P → Q)	A (aim for (P → R) and use CP)
3	3. P	A (aim for R and use CP)
2,3	4. Q	3,2 MPP
1,2,3	5. R	4,1 MPP
1,2	6. (P → R)	3,5 CP
1	7. ((P → Q) → (P → R))	2,6 CP
	8. (Q → R) → ((P → Q) → (P → R))	1,7 CP

(b) |- (P → (Q → (P & Q)))

1	1. P	A (aim for (Q → (P & Q)) and use CP)
2	2. Q	A (aim for P&Q and use CP)
1	3. (P & Q)	1,2 &I
1	4. (Q → (P & Q))	2,3 CP
	5. (P → (Q → (P & Q)))	1,4 CP

(parts of problems on p. 42 discussed, other general questions).

Meeting 11

Exam I (in class)

Lecture 12

Return exams., hand out solutions.

Scale: 90↑ A
80↑ B
70↑ C
50↑ D

Most people did very well. This is good, because we will build on this material. In particular, when we take up predicate logic later, we will take rules of proof and add to it. If you are still unsure about the proof system, I would be happy to work through further problems with you.

Questions ?

We have a set of rules of proof for propositional logic. We would like to say that the rules have the following two desirable features:

Soundness: If $F_1,...,F_n \vdash G$, then whenever $F_1,...,F_n$ are all true, G must be true.

Completeness: If G is true whenever $F_1,...,F_n$ are all true, then $F_1,...,F_n \vdash G$.

We have talked a little about soundness. What we decided was that we could not explain soundness without pinning down truth. Similarly, we cannot explain completeness.

Putting some restrictions on the notion of truth, we will be in a position to explain why our set of rules is sound and complete.

Recall that for any wff F, we have a finite sequence

$G_1,...,G_r$ such that $G_r = F$, and for each G_i, one of the following holds:

(i) G_i is a propositional variable,
(ii) $G_i = \neg G_k$ for some $k < i$,
(iii) $G_i = (G_k \_ G_j)$, where $j, k < i$ and _ is one of &, ∨, or →.

We call such a sequence a formation sequence for F.

Example: Let F = ((P ∨ (¬Q & ¬P))

We have the following formation sequence for F:
P, Q, ¬Q, ¬P, (¬Q & ¬P), (P ∨ (¬P & ¬P))

We shall give rules which allow us to calculate the truth-value for F, given truth values for P and Q. These rules ignore possibilities which are present in English and other natural languages. They have the virtue of being precise.

Rules for computing truth-values.

(1) ¬F is true if and only if F is false
(2) (F & G) is true if and only if F and G are both true
(3) (F ∨ G) is true if and only if at least one of F, G is true
(4) (F → G) is true unless F is true and G is false

We are treating ↔ not as a basic connective, but as an abbreviation. Nonetheless, we could add a clause.

(5) (F ↔ G) is true if and only if F and G have the same truth value (i.e., both are true or both are false).

Do these rules seem natural ? Let us see what rules say when we give interpretations to propositional variables.

Example: Let P--it is Monday, Q--the sun is shining.

(a) Note that (P ∨ Q) is true if it is a sunny Monday.
We have (P ∨ Q) true if P is true or Q is true, possibly both are true.

(b) Note that (P → Q) is true if it is Tuesday and sunny, or Tuesday and overcast.
We have (P → Q) true except when it Monday and the sun fails to shine.

Returning to the example above, let us calculate the truth value for (P ∨ (¬P & ¬P)), assuming that P is false and Q is true. We follow the formation sequence, calculating truth values as we go.

P	Q	¬Q	¬P	(¬Q & ¬P)	(P ∨ (¬P & ¬P))
F	T	F	T	T	T

We could consider all possible assignments of truth values to the propositional variables, forming a truth table.

P	Q	¬Q	¬P	(¬Q & ¬P)	(P ∨ (¬P & ¬P))
F	F	T	T	T	T
F	T	F	T	F	F
T	F	T	F	F	T
T	T	F	F	F	T

Thus, (P ∨ (¬P & ¬P)) is true unless P is false and Q is true.

Definitions: A wff is tautologous if it is always true (on all lines of its truth table). It is inconsistent if it is never true. It is contingent if it is sometimes true and sometimes false; i.e., true on at least one line and false on at least one line.

The example above is contingent.

Let us look at further examples.

1. (P ∨ ¬P)

P	¬P	(P ∨ ¬P)
F	T	T
T	F	T

From the truth table, we see that this wff is tautologous.

2. (P & ¬P)

P	¬P	(P & ¬P)
F	T	F
T	F	F

This is inconsistent.

2. (P → (Q → R))

P	Q	R	(Q → R)	(P → (Q → R))
F	F	F	T	T
F	F	T	T	T
F	T	F	F	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

This is contingent.

The method of truth tables is perfectly general. It lets us determine whether any given wff is tautologous, contingent, or inconsistent.

In the last example, and in some other examples, especially those involving implications, we may find a short-cut. We have (P → (Q → R)) false just in case P is true (Q → R) is false, and this happens just in case Q is true and R is false. In all, the statement is false just in case P and Q are true while R is false.

Homework #8: p. 73, 1 a,b,c,d,e,f,g,h,i,j,k

Solutions to #8
p. 73, 1 a,b,c,d,e,f,g,h,i,j,k

(a) (P → P) is tautologous

P	(P → P)
F	T
T	T

(b) (P → ¬P) is contingent

P	¬P	(P → ¬P)
F	T	T
T	F	F

(c) ¬(P → P) is inconsistent

P	(P → P)	¬(P → P)
F	T	F
T	T	F

(d) P is contingent

P
F
T

(e) (¬P → (P → Q)) is tautologous

P	Q	¬P	(P → Q)	(¬P → (P → Q))
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	F	T	T

(f) ((P ↔ Q) ↔ ¬(P ↔ ¬Q)) is tautologous

Note: (F ↔ G) is true if and only if F and G have the same truth value. We can check this, thinking of (F ↔ G) as ((F → G) & (G → F)).

P	Q	¬Q	(P ↔ Q)	(P ↔ ¬Q)	¬(P ↔ ¬Q)	((P ↔ Q) ↔ ¬(P ↔ ¬Q))
F	F	T	T	F	T	T
F	T	F	F	T	F	T
T	F	T	F	T	F	T
T	T	F	T	F	T	T

(g) ((P & Q) & ¬(P ↔ Q)) is inconsistent

P	Q	(P & Q)	(P ↔ Q)	¬(P ↔ Q)	((P & Q) & ¬(P ↔ Q))
F	F	F	T	F	F
F	T	F	F	T	F
T	F	F	F	T	F
T	T	T	T	F	F

(h) ((P ∨ ¬Q) & ¬(¬P → ¬Q)) is inconsistent

P	Q	¬P	¬Q	(P ∨ ¬Q)	(¬P → ¬Q)	¬(¬P → ¬Q)	((P ∨ ¬Q) & ¬(¬P → ¬Q))
F	F	T	T	T	T	F	F
F	T	T	F	F	F	T	F
T	F	F	T	T	T	F	F
T	T	F	F	T	T	F	F

(i) $((P \& Q) \to R)) \to ((P \to R) \& (Q \to R)))$ is contingent

PQR	(P&Q)	((P&Q)→R)	(P→R)	(Q→R)	((P→R)&(Q→R))	(((P&Q)→R))→((P→R)&(Q→R)))
FFF	F	T	T	T	T	T
FFT	F	T	T	T	T	T
FTF	F	T	T	F	F	F
FTT	F	T	T	T	T	T
TFF	F	T	F	T	F	F
TFT	F	T	T	T	T	T
TTF	T	F	F	F	F	T
TTT	T	T	T	T	T	T

(j) $(((P \vee Q) \to R) \leftrightarrow ((P \to R) \& (Q \to R)))$ is tautologous

P Q R	(PvQ)	((PvQ)→R)	(P→R)	(Q→R)	((P→R)&(Q→R))	(((PvQ)→R)↔((P→R)&(Q→R)))
F F F	F	T	T	T	T	T
F F T	F	T	T	T	T	T
F T F	T	F	T	F	F	T
F T T	T	T	T	T	T	T
T F F	T	F	F	T	F	T
T F T	T	T	T	T	T	T
T T F	T	F	F	F	F	T
T T T	T	T	T	T	T	T

(k) $(((P \to Q) \& (R \to S)) \to ((P \vee R) \to (Q \vee S)))$ is tautologous.

Instead of writing out the whole truth table, we argue as follows. The full statement can only be false if $((P \to Q) \& (R \to S))$ is true while $((P \vee R) \to (Q \vee S))$ is false. Then $(P \vee R)$ must be true while $(Q \vee S)$ is false, which means that Q and S are both false and either P is true or R is true. If P is true and Q is false, then $(P \to Q)$ is false, and if R is true and S is false, then $(R \to S)$ is false, so in either case, $((P \to Q) \& (R \to S))$ is false. Therefore, there is no assignment of truth values to P, Q, R, and S which makes the statement false.

Lecture 13

We can use truth tables to determine whether a wff is always true. We can also use truth tables to analyze arguments for soundness.

Definition: We say that $F_1,...,F_n$ logically imply G if G is true whenever $F_1,...,F_n$ are all true. We write $F_1,...,F_n \models G$ if $F_1,...,F_n$ logically imply G.

Compare notation $F_1,...,F_n \vdash G$, which means that there is a proof of G from $F_1,...,F_n$.

To determine whether $F_1,...,F_n \models G$, we have the following method. We write the truth tables for the formulas all at once, and see if on the lines where $F_1,...,F_n$ are all true, G is also true.

Example 1: Use truth tables to determine whether $(P \,\&\, Q) \models (P \vee Q)$.

P	Q	(P & Q)	(P ∨ Q)
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

The assumption (P & Q) is only true on the last line, and there the conclusion (P ∨ Q) is also true, so we have $(P \,\&\, Q) \models (P \vee Q)$.

Example 2: Show that the following logical implication fails. Give a specific assignment of truth-values to the propositional variables which makes the premises true and the conclusion false.

$(P \rightarrow R), (Q \rightarrow \neg R) \not\models \neg(P \vee Q)$

P	Q	R	(P → R)	¬R	(Q → ¬R)	(P ∨ Q)	¬(P ∨ Q)
F	F	F	T	T	T	F	T
F	F	T	T	F	T	F	T
F	T	F	**T**	T	**T**	T	**F**

If P and R are false and Q is true, then the hypotheses are true and the conclusion is false.

We may analyze arguments phrased in English. In fact, some people in law school have indicated that they use this method.

Example 3: Recall argument from first day of class.

Premises: If the river floods, then our entire wheat crop will be destroyed.

The river will flood if there is an early thaw.

In any case, there will be heavy rains later in the summer.

Conclusion: If there is an early thaw, our entire community will be bankrupt unless there are heavy rains later in the summer.

Let P--the river floods, Q--wheat crop destroyed, R--early thaw, S--heavy rains later in summer, T--community bankrupt

We translated argument.
Premises: $(P \rightarrow Q)$, $(R \rightarrow P)$, S
Conclusion: $(R \rightarrow (\neg S \rightarrow T))$

Do the premises logically imply the conclusion; i.e., do we have
$(P \rightarrow Q), (R \rightarrow P), S \models (R \rightarrow (\neg S \rightarrow T))$?

P	Q	R	S	T	$(P \rightarrow Q)$	$(R \rightarrow P)$	$\neg S$	$(\neg S \rightarrow T)$	$(R \rightarrow (\neg S \rightarrow T))$

We could write out the full truth table, an unpleasant prospect (with 32 lines). Thinking about the truth table, we look for a short-cut. The conclusion $(R \rightarrow (\neg S \rightarrow T))$ can only fail if R is true and $(\neg S \rightarrow T)$ is false. Then $\neg S$ is true and T is false. Then S is false, contradicting one of the premises. Therefore, whenever the premises are all true, the conclusion is also true.

Homework #9 (the first two problems are from pp. 82-83):

1. Show using truth tables that the following logical implications hold.

(a) (P & Q) |= (P → Q)
(b) (¬P & Q) |= (P → Q)
(c) (¬P & ¬Q) |= (P → Q)

2. Show that the following logical implications fail. Give a specific assignment of truth-values to the propositional variables which makes the premises all true and the conclusion false.

(a) ((P & Q) → R) |≠ (P ∅ R)
(b) (P → (Q v R)) |≠ (P ∅ Q)
(c) (P → Q), (P → R) |≠ (Q ∅ R)

3. (a) Translate the argument below.

Tom embezzled money from ABAG. If he is caught, then he will go to jail and his assistant will be fired. Tom's assistant is being fired. Therefore, Tom is going to jail.

In your translation, let E be "Tom embezzled money from ABAG", let C be "Tom is caught", let J be "Tom goes to jail", and let A be "the assistant is fired".

(b) Indicate which are the assumptions and which is the conclusion.

(c) Determine whether the argument is sound. Justify your answer by our current methods, either explaining why, in the full truth table, whenever the assumptions are all true, the conclusion is true, or else giving an assignment of truth values to E, C, J, A which makes the assumptions true and the conclusion false.

Review problems (proofs)

1. P, (P → Q) |- (P & Q)
2. ¬P, (Q → P) |- (¬P & ¬Q)
3. |- (P → (P ∨ Q))
4. (P → Q), (Q → R) |- (P → R)
5. P, (R → ¬P) |- ¬R
6. (R → P), (R → ¬P) |- ¬R
7. (P ∨ Q), (P → R), (Q → R) |- R

Solutions to #9

1. (a) (P & Q) |= (P → Q)

P	Q	(P & Q)	(P → Q)
F	F	F	T
F	T	F	T
T	F	F	F
T	T	**T**	**T**

(b) (¬P & Q) |= (P → Q)

P	Q	¬P	(¬P & Q)	(P → Q)
F	F	T	F	T
F	T	T	**T**	**T**
T	F	F	F	F
T	T	F	F	T

(c) (¬P & ¬Q) |= (P → Q)

P	Q	¬P	¬Q	(¬P & ¬Q)	(P → Q)
F	F	T	T	**T**	**T**
F	T	T	F	F	T
T	F	F	T	F	F
T	T	F	F	F	T

2. (a) ((P & Q) → R) |≠ (P ∅ R)

P	Q	R	(P & Q)	((P & Q) → R)	(P → R)
T	F	F	F	T	F

We have (P → R) false if P is true and R is false. Given this, ((P & Q) → R) is true if Q is false. So, let P be true, Q and R false.

(b) (P → (Q v R)) |≠ (P ∅ Q)

P	Q	R	(Q v R)	(P → (Q v R))	(P → Q)
T	F	T	T	T	F

We have (P → Q) false if P is true and Q is false. Given this, (P → (Q v R)) is true if R is true. So, let P be true, Q false, and R true.

(c) (P → Q), (P → R) |≠ (Q ∅ R)

P	Q	R	(P → Q)	(P → R)	(Q → R)
F	T	F	T	T	F

We have (Q → R) false if Q is true and R is false. Given this, (P → Q) is true, regardless of P, and (P → R) is true if P is false. So let P be false, Q true, and R false.

3. Tom embezzled money from ABAG. If he is caught, then he will go to jail and his assistant will be fired. Tom's assistant is being fired. Therefore, Tom is going to jail. Translation, letting E be "Tom embezzled money from ABAG", let C be "Tom is caught", let J be "Tom goes to jail", and let A be "the assistant is fired":

Premises:
1. E
2. (C → (J & A))
3. A

Conclusion: J

E	C	J	A	(J & A)	(C → (J & A))
T	F	F	T	F	T

Let E and A be true and let J be false. Given this, (C → (J & A)) is true provided that C is false. Then for E and A true and C and J false, the premises are all true while the conclusion is false. Therefore, the argument is unsound.

Lecture 14

Hand out homework and solutions to prev. assignment.

Questions on current homework ?

For Monday, I want you to begin on papers. Make a tentative decision about a topic. You may write joint papers. Recommended number of authors: two.

Should be on some aspect of logic not fully developed in class. Approximately 5 pp. (possibly longer if joint). Choose a topic that interests you, and that seems likely to interest your classmates. The paper will be due just before Easter, and later you will give a short talk in class, based on the paper.

Possible topics for papers, with some references

1. Logical Paradoxes

Kleene, Introduction to Metamathematics, Chapter III. [This book is technical, but contains interesting material.]

Quine, "Paradox", Scientific American , vol. 206(1962).

2. What is and is not provable, using "natural" assumptions ?

Dawson, Logical Dilemmas: The Life and Work of Kurt Gö del , Peters, 1997 (new book, should be out any day).

[Gö del is responsible for Completeness. He also proved an "Incompleteness Theorem" saying that no recognizable set of true assumptions is enough to prove all of the statements true about the natural numbers. In addition, he isolated the Continuum Hypothesis, an important statement about sizes of subsets of the real numbers, that cannot be proved or disproved using the usual assumptions.]

(a) Geometry and the parallel postulate, non-Euclidean geometries.

[Showing that the parallel postulate in plane geometry cannot be proved from the other postulates, by describing settings in which the other postulates are true and the parallel postulate fails.]

Davis and Hersch, The Mathematical Experience, pp. 217-222.

Penrose, "The geometry of the universe", in Mathematics Today: Twelve Informal Essays, ed. by L. Steen, pp. 83-94.

(b) Number theory, Gö del's "Incompleteness" Theorem.

The idea in the proof of Gö del's Incompleteness Theorem is related to the Liar Paradox. There is a sentence which refers to itself and has the meaning "I am unprovable."

Article in Scientific American, June, 1956.

Nagel and Newman, Gö del's Proof, New York Univ. Press, 1958.

Smullyan, What is the Name of this Book ?, also The Lady or the Tiger. [These books contain mathematical puzzles involving self reference.]

Set theory and the "Continuum Hypothesis" (Gö del and Cohen), requires coordination with someone working on #3

[The Continuum Hypothesis says that every subset of the reals which is not countable is the same size as the full set of reals. Cantor tried unsuccessfully to prove this. Gö del showed that the negation cannot be proved from the usual assumptions about sets (assuming that these are consistent). Cohen showed that the statement cannot be proved.

Cohen and Hersch, "Non-Cantorian Set Theory", Scientific American, vol. 217(1967), pp. 104-117.

Davis and Hersch, The Mathematical Experience , pp. 223-236.

Gö del, "What is Cantor's continuum problem ?" The American Mathematical Monthly, vol. 54(1947), pp. 515-525.

3. Set size, comparing sizes of infinite sets

Kleene, Introduction to Metamathematics , Chapter I. [Technical]

Reid, From Zero to Infinity (especially last chapter). [Non-technical]

Steen, S., Mathematics: The Man-Made Universe , Chapter 18.

4. What can be done by machines ? (related to #2 (b))

(a) What is computable ? (definitions by Turing, Gö del and others)

[Turing responsible for best definition of what machine can do, idea of program. Leader of British group that broke German codes during World War II.]

Hodges, The Enigma (biography of Alan Turing). [Author is mathematician and gay rights activist.]

Davis, "What is a computation ?" article in Mathematics Today: Twelve Informal Essays, ed. by L. Steen.

Yasuhara, Recursive Function Theory and Logic, Chapter 1 (what is a Turing machine ?)

(b) Unsolvability of Hilbert's 10th Problem

[Hilbert, in 1900, gave a talk describing 23 problems. This list has shaped a great deal of 20th century mathematics. The 10th problem was to give an algorithm, or procedure, for deciding whether a given polynomial with integer coefficients has integer roots. The problem turned out to be unsolvable. Julia Robinson, an American, was responsible for preliminary work. Final work was by a Russian, Yuri Matijasevich.

Reid, several different short biographies of J. Robinson (Robinson and Reid were sisters).

Yasuhara, Recursive Function Theory and Logic, Chapter 1, especially, p. 35.

(c) Proof by machine

[Machines have been used in the proofs of certain theorems. Some people find this disconcerting. They want to understand the "idea" behind the proof, or they suspect machine errors. There was a famous problem on the least number of colors which suffice to color maps in the plane, so that no two countries with a common border have the same color. Appel and Haken solved the problem, making some reductions and then using a computer to analyze a large number of cases.]

"The four-color problem", Appel and Haken, in Mathematics Today: Twelve Informal Essays, ed. by L. Steen.

Steen, S., Mathematics: the Man-Made Universe, Chapter 14 (on "4-color problem", written before problem was solved).

Davis and Hersh, The Mathematical Experience, pp. 380-386 (on Appel and Haken's solution of 4-color problem).

Kolata, article in New York Times, Dec. 10, 1996, on very recent proof by machine.

5. Problems which are "hard" but not unsolvable

[Determining whether a wff is tautologous is an example of a problem for which there is an algorithm, but which, as the number of propositional variables grows, becomes practically out of reach. There are other problems of an applied nature which present exactly the same difficulties.'

Graham, Combinatorial scheduling theory, in Mathematics Today: Twelve Informal Essays, ed. by L. Steen.

6. Infinitesimal numbers (A. Robinson)

[The originators of calculus found it useful to imagine infinitely small numbers, while admitting that there were no such real numbers. Abraham Robinson enlarged the number system to include "infinitesimals", retaining important properties of the reals].

Davis and Hersch, The Mathematical Experience, pp. 237-254.
Keisler, Calculus.

7. Non-classical logic, Intuitionism

Heyting, Intuitionism.

Kleene, Introduction to Metamathematics, pp. 46-53 [reference to Fermat's Last Theorem is out of date, as the theorem has recently been proved].

8. Derived rules of proof

Reference: Lemmon, Beginning Logic.

9. Soundness and Completeness

Reference: Lemmon, Beginning Logic.

Lecture 15

Hand out homework and solutions, material on papers.
I will say a little about Soundness and Completeness for Propositional Logic. On Wednesday, or Friday at the latest, we will start on Predicate Logic.

In the notation we have introduced, Soundness and Completeness may be stated as follows.

<u>Soundness</u>: If $F_1,...,F_n$ |- G, then $F_1,...,F_n$ |= G.

<u>Completeness</u>: If $F_1,...,F_n$ |= G, then $F_1,...,F_n$ |- G.

Consider Soundness first. I won't give complete proof, but the idea is to show that for each step in a proof of G from $F_1,...,F_n$, the statement is logically implied by the indicated assumptions.

Begin with some basic facts, corresponding to different rules.

A: F |= F

MPP: F, (F → G) |= G

MTT: (F → G),¬G |= ¬F

CP: If F,G |= H, then F |= (G → H) (could replace F by tuple)

DN: (a) F |= ¬¬F
(b) ¬¬F |= F

&I: F, G |= (F & G)

&E: (a) (F & G) |= F
(b) (F & G) |= G
vI: (a) F |= (F v G)

vE: If F,G_1 |= H and F,G_2 |= H, then F,(G_1vG_2) |= H (could replace F by tuple)

RAA: If F,G |= (H & ¬H), then F |= ¬G (could replace F by tuple)

<u>Example 1</u>: (P → Q), (Q → R) |- (P → R)

1	1. (P → Q)		A
2	2. (Q → R)		A
3	3. P		A
1,3	4. Q	3,1	MPP

1,2,3	5. R	4,2	MPP	
1,2	6. (P → R)	3,5	CP	

Check that for each line, the statement is true whenever the indicated assumptions are all true.

<u>Lines,1, 2, 3</u>: These lines use rule A.

<u>Lines 4 and 5</u>: These lines use rule MPP.

We have (P → Q), P |= Q. Also, (Q → R), Q |= R, so P, (P → Q), (Q → R) |= R.

<u>Line 6</u>: This line uses rule CP.

From line 5, (P → Q), (Q → R), P |= R. Therefore, (P → Q), (Q → R) |= (P → R).

<u>Example 2</u>: P, (R → ¬P) |- ¬R

1	1. P	A	
2	2. (R → ¬P)	A	
3	3. R	A	
2,3	4. ¬P 3,2	MPP	
1,2,3	5. (P & ¬P)	1,4	&I
1,2	6. ¬R 3,5	RAA	

<u>Claim</u>: For each line, the current statement is true whenever the indicated assumptions are true.

Lines 1,2,3 use rule A, so statement is clear.
Line 4 uses rule MPP, so statement follows as in Example 1.
Line 5 uses rule &I.

Line 6 uses rule RAA.

<u>Basic Fact</u>: If $F_1,...,F_n$,F |= (G & ¬G), then $F_1,...,F_n$ |= ¬F.

Since (G & ¬G) is inconsistent, we can <u>never</u> have F_1, ..., F_n, F all true. So, whenever $F_1,...,F_n$ are all true, F must be false.
From line 5, we have P, (R → ¬P), R |= (P & ¬P). Therefore, P, (R → ¬P) |= ¬R.

<u>Example 3:</u> (P ∨ Q), (P → R), (Q → R) |- R

1	1. (P ∨ Q)	A
2	2. (P → Q)	A
3	3. (Q → R)	A

4	4.	P		A
2,4	5.	Q	4,2	MPP
2,3,4	6.	R	5,3	MPP
7	7.	Q		A
3,7	8.	R	7,3	MPP
1,2,3	9.	R	1,4,6,7,8	∨E

Lines 1-8 are by A, MPP rules.
Line 9 uses ∨E.
From line 6, we have $(P \to Q), (Q \to R), P \models R$ and from line 8, we have $(Q \to R), Q \models R$.
Therefore, $(P \to Q),(Q \to R),(P \vee Q) \models R$.

With this much discussion of Soundness, turn to Completeness.

Homework #10

A. On her 100th birthday, sweet lady is asked, "What is the secret of your long life ?" "Oh", she answers, "I strictly follow my diet, " and proceeds to give the following argument.

If I don't drink beer for dinner, then I always have fish. Any time I have both beer and fish for dinner, then I do without ice cream. So, if I have ice cream or don't have beer, then I never eat fish.

(1) Let B--beer, F--fish, I--ice cream.

Premises: 1. $(\neg B \rightarrow F)$

2. $((B \,\&\, F) \rightarrow \neg I)$

Conclusion: $((I \vee \neg B) \rightarrow \neg F)$

(2) The conclusion fails if F is true and either I is true or B is false. Suppose F is true. Then 1 must be true, and 2 is true if either B or I is false. If B is false and F is true, the premises are true and the conclusion is false, regardless of I, so the argument is unsound.

B. (a) Naive realism leads to physics, and physics, if true, shows that naive realism is false. Therefore, naive realism, if true, is false, therefore, it is false.

Let R--naive realism, P--Physics.

Premises: 1. $(R \rightarrow P)$

2. $(P \rightarrow \neg R)$

Conclusions: $(R \rightarrow \neg R)$, $\neg R$

R	P	$(R \rightarrow P)$	$\neg R$	$(P \rightarrow \neg R)$	$(R \rightarrow \neg R)$
F	F	**T**	**T**	**T**	**T**
F	T	**T**	**T**	**T**	**T**
T	F	F	F	T	T
T	T	T	F	F	F

The argument is sound. When the premises are true, so are the conclusions.

(b) The derivative of f is given by f'(x) = 3x(x+2). Hence, f'(x) > 0 if both x > 0 and x+2 > 0 or if both x < 0 and x+2 < 0. Thus, f'(x) > 0 if x > 0 or if x < -2.

Let P--f'(x) = 3x(x+2), Q--f'(x) > 0, R_1--x > 0, R_2--x < 0, S_1--x+2 > 0, S_2--x+2 < 0, T_1--x > -2, T_2--x < -2.

Premises: 1. P

2. $(P \rightarrow (((R_1 \,\&\, S_1) \vee (R_2 \,\&\, S_2)) \rightarrow Q))$

(tacit) 3. $(S_1 \leftrightarrow T_1)$

4. $(S_2 \leftrightarrow T_2)$

5. $(R_1 \rightarrow T_1)$

6. $(T_2 \rightarrow R_2)$

Conclusion: $((R_1 \vee T_2) \rightarrow Q)$

The conclusion only fails if Q is false and either R_1 or T_2 is true. Then if 1 holds, P is true. If R_1 is true, then T_1 and S_1 are both true, so 2 fails. If T_2 is true, then R_2 and S_2 are both true, so again 2 fails. If the premises are all true, then so is the conclusion. The argument is sound.

(c) If God were all-good, He would want to create a world without evil; while if He were all-powerful, He could create such a world. Yet evil exists. Thus, God cannot be both all-good and omnipotent.
Let G--all-good, W--desires world with no evil, P--all-powerful, C--can guarantee no evil, E--evil exists

Premises: 1. $(G \rightarrow W)$
2. $(P \rightarrow C)$
3. E
tacit: 4. $((W \& C) \rightarrow \neg E)$
Conclusion: $\neg(G \& P)$

The conclusion only fails if G and P are both true. Then if 1, 2, and 4 hold, W and C are true, and E is false, so 3 fails. If the premises are all true, then so is the conclusion. The argument is sound.

(d) According to Steiglitz's Theorem, the null radical, if raised to a power of 2n, would have a harmonic kernel. But it doesn't, even though I've raised the radical to that power. The crucial Lemma 3 upon which the theorem depends must therefore be false.
Let S--Steiglitz's Theorem holds, N--null radical raised to power of 2n, H--kernel is harmonic, L--Lemma 3 holds.

Premises: 1. $(S \rightarrow (N \rightarrow H))$
2. $(\neg H \& N)$
tacit: 3. $(L \rightarrow S)$
Conclusion: $\neg L$

If the conclusion fails, then L is false. If 2 holds, H is false and N is true. Then if 1 holds, S is false, while if 3 holds, S is true. If the premises are all true, then so is the conclusion. Therefore, the argument is sound.

(e) If there is justice in this life, no after-life is necessary. On the other hand, if there is no justice in this life, then one has no reason to believe God is just. However, if one has no reason to believe that, what reason do we have to think He'll provide us with an after-life ? So, either no after-life is needed or else we have no reason to think God will provide one.
Let J--justice in this life, N--after-life needed, R--reason to believe God is just, P--reason to after-life provided

Premises: 1. $(J \rightarrow \neg N)$
2. $(\neg J \rightarrow \neg R)$
3. $(\neg R \rightarrow \neg P)$
Conclusion: $(\neg N \vee \neg P)$

The conclusion only fails if N and P are both true. Then if 1 and 2 hold, J is false and R is false. Then 3 fails. If the premises are all true, then so is the conclusion. Therefore, the argument is sound.

Lecture 16

Collect homework, hand out solutions to #11 and new problems.

Note on papers: Grade will be based on the following three criteria.

1. Mathematical content (giving precise statements, correct explanations, etc.)
2. Ambition (making use of difficult references, or combining a number of references, etc.)
3. Presentation (well-organized, well-written)

Discuss at least first problem (left hanging last time).

I indicated why Soundness holds. Now, say something about Completeness.

Completeness: If $F_1,...,F_n \models G$, then $F_1,...,F_n \vdash G$.

Consider special case.

Special Case: If $\models G$, then $\vdash G$.

Claim: Special case implies general case.

Suppose $F_1, ..., F_n \models G$.

Then $\models ((F_1 \& ... \& F_n) \rightarrow G)$, by the definition of satisfaction.

Assuming special case, we have $\vdash ((F_1 \& ... \& F_n) \rightarrow G)$.

Then $F_1, ..., F_n \vdash G$. We use &I, MPP.

So, the special case implies the general case.

To see why special case holds, consider very special case, where G has just one propositional variable P.

Two steps:

I. Show that for all G built up from just P, $P \vdash G$ or $P \vdash \neg G$, and $\neg P \vdash G$ or $\neg P \vdash \neg G$.

II. Show that $\vdash (P \vee \neg P)$.

Recall (we did this earlier, but it was a bit tricky).

1	1. ¬(P ∨ ¬P)	A (use RAA; try to show ¬P)
2	2. P	A (use RAA)
2	3. (P ∨ ¬P)	2 ∨I
1,2	4. ((P ∨ ¬P) & ¬(P ∨ ¬P))	1,3 &I (as planned in line 2)
1	5. ¬P	2,4 RAA
1	6. (P ∨ ¬P)	5 ∨I
1	7. ((P ∨ ¬P) & ¬(P ∨ ¬P))	1,6 &I (as planned in line 1)
	8. ¬¬(P ∨ ¬P)	1,7 RAA
	9. (P ∨ ¬P)	

III. Complete proof for G build up from just P, as follows.

Supposing that |= G, we try to show that (P ∨ ¬P) |- G. Then taking the proof of (P ∨ ¬P) from no assumptions, and the proof of G from (P ∨ ¬P), we get a proof of G from no assumptions.

To show that (P ∨ ¬P) |- G, we first show that P |- G.

By I, either P |- G or P |- ¬G. If P |- ¬G, then by Soundness, we would have P |= ¬G. This is impossible, since G is always true. Therefore, P |- G.

Next, by a similar argument, we show that ¬P |- G.

Then using ∨E, we have (P ∨ ¬P) |- G.

For G involving two or more propositional variables, the idea is the same.

I. For all G built up from P and Q, ±P,±Q |- ±G.

II. |- ((P&Q) ∨ (P&¬Q)) ∨ ((¬P&Q) ∨ (¬P&¬Q)).

III. Combine I, II, Soundness to show that if |= G, then |- G.

For G involving two variables P and Q, you have done exercises toward step I. Exercises in direction of step II. Step III is as for single variable.

Solutions to #11

1. (a) You say it didn't rain last night, eh ? Well, if it didn't rain, the sidewalks wouldn't be wet. But they're soaked, so looks like it didn't <u>not</u> rain either !
Letting R--it rained, S--sidewalks are wet, we may translate: $(\neg R \rightarrow \neg S), S \vdash \neg\neg R$.

The argument is sound. We could give a proof. Or, we may argue that
$(\neg R \rightarrow \neg S), S \models \neg\neg R$; i.e., whenever the premises are true, the conclusion is true, or whenever the conclusion fails, some premise must fail.

(b) Olga must have played this year. If she hadn't, we'd have been without a decent goalie. But it says here in the Newsletter that we placed our goalie on the All-League Team, which means she had to be pretty good.
Letting O--Olga played, G--goalie was good, A--goalie on All-League Team, we may translate: $(\neg O \rightarrow \neg G), A, (A \rightarrow G) \vdash O$. The argument is sound. We could give a proof, or argue in terms of truth.

(c) This levy fails and the schools will just have to close. You couldn't even pay the teachers. They aren't going to work for nothing, you know. Would you ? No, if the schools are going to stay open, you'll need money at least for salaries.
Letting L--school levy passes, C--schools close, P--teachers are paid, W--teachers work, we may translate: $(\neg L \rightarrow \neg P), (\neg P \rightarrow \neg W), (\neg C \rightarrow P) \vdash \neg L \rightarrow C$. The argument is sound. We could give a proof, or argue in terms of truth.

(d) "Me? I always get what I want. If I can't talk that fool, Arlo P. Frozzlbottom, into a large donation, I'll be a monkey's uncle. I hear he's a real sucker. Say, what'd you say your name was ? "Frozzlbottom...Arlo P." "Spare a banana ?"
Letting D--I can talk F. into donation, M--I'll be a monkey's uncle, we may translate:
$(\neg D \rightarrow M), \neg D \vdash M$. The argument is sound. We could give a proof, or argue in terms of truth.

(e) Lousy reporters can't do anything right ! Let Claghorn put his foot in it and what do they do ? --they get every word. If he's in trouble, it's only because they didn't have the decency to misquote him as usual. A rotten deal if ever I saw one. The Senator's statement wouldn't have caused him the least embarrassment if he just hadn't been quoted correctly.
Letting R--reporters are no help, C--Claghorn said something stupid, Q--quoted correctely, E--embarrassed, we may translate: $(C \rightarrow Q), (\neg Q \rightarrow \neg E) \vdash (\neg Q \rightarrow \neg(C \rightarrow E))$.
The argument is sound. We could give a proof, or argue in terms of truth.

2. (a) $(P \rightarrow Q), (R \rightarrow Q) \nvdash (P \varnothing R)$
If P and Q are true and R is false, then the premises hold and the conclusion fails.

(b) $\neg P, Q \nvdash \neg(P \rightarrow Q)$
If P is false and Q is true, then $(P \rightarrow Q)$ is true so $\neg(P \rightarrow Q)$ is false.

(c) $\neg(P \rightarrow Q), (Q \rightarrow R) \vdash \neg(P \rightarrow R)$
If P and R are true and Q is false, then the premises hold and the conclusion fails.

3. (a) Suppose the President signs it. He'll take a lot of heat from the liberals. Too much, I'm afraid. He won't risk that kind of heat--not if he wants to win in the primary. He'll sign it only if he doesn't want to get re-elected.
Letting S--signs, H--takes heat, W--wants to win, we may translate:
$(S \rightarrow H), (W \rightarrow \neg H) \vdash (S \rightarrow \neg W)$.
The argument is sound. We could justify this either of two ways.

(b) Fritz won't go to the party unless Emily or Mary is there. So only if Mary shows will he attend, because you can be certain Emily'll go only if Fritz doesn't.
Leting F--Fritz goes, E--Emily goes, M--Mary goes, we translate:
$(F \rightarrow (E \vee M)), (E \rightarrow \neg F) \vdash (F \rightarrow M)$.
The argument is sound. We could justify this in either of two ways.

(c) Looks like Anne and Sue won't get to see each other. Neither Bill nor Sue can make it; nor can Anne and Chuck. A real shame... the only opportunity the two had to get together.

Letting B--Bill comes, S--Sue comes, A Anne and Chuck come, M--Anne and Sue meet, we translate: $\neg(B \vee S), \neg A, (M \rightarrow (A \,\&\, S)) \vdash \neg M$. The argument is sound. We could justify this in either of two ways.

(d) Election-year politicking is enough to make one cry. Neither the White House nor Congress is going to budge on tax reform unless one of them compromises on energy. And Congress sure isn't going to do any compromising--not in an election-year. There you have it: the White House gives in on energy--and you know what that means, good-by energy policy ! --or else it'll be yet another year with no tax relief.

Letting L--election year, T--tax relief, E--energy policy, TC--compromise on taxes, EC--compromise on energy, we may translate:
$L, (TC \rightarrow EC), (L \rightarrow \neg(TC \vee EC)), (T \rightarrow TC), (EC \rightarrow \neg E) \vdash (\neg E \vee \neg T)$. The argument is sound. We may justify this in either of two ways.

Lecture 17

Collect homework and discuss new translation problems.

We will use Soundness and Completeness.

If you want to show that $F_1,...,F_n \models G$, you could give the truth table. Or, you could give a proof of G from $F_1,...,F_n$. If you want to determine whether $F_1,...,F_n \vdash G$, you could use truth tables, or think in terms of the definition of truth, to check whether $F_1,...,F_n \models G$. In some cases, giving a proof is easier than writing out the full truth table.

Question: What can a machine do ?

A machine can, in principle, determine whether a proposed argument, in propositional logic, is sound (constructing truth table). Could a machine, in principle, construct a proof ? There is a not very satisfying method. Start writing down all possible finite sequences of wff's, checking whether they are proofs. If there is a proof, we will come to it eventually.

I said we would start today on predicate logic.

Consider the following arguments.

Example 1 (famous)

Premises: 1. All men are mortal.
2. Socrates is a man.

Conclusion: Socrates is mortal.

Example 2 (not famous, but of the same form)

Premises: 1. All dogs have fleas.
2. Rocky is a dog.

Conclusion: Rocky has fleas.

Let us try to translate Example 1 into propositional logic

Let A--All men are mortal, S--Socrates is a man, M--Socrates is mortal (no connectives inside any of these statements). Then A and S are the premises and M is the conclusion.

Example 2 is the same, where A--all dogs have fleas, etc.

As translated, the arguments are unsound.

But, there is something believable about these argument, something the translation failed to capture.

Propositional logic doesn't provide a way to talk about properties like being a man or a dog. It doesn't have names like Socrates or Rocky, for distinguished objects. There is no way to say something holds for all objects of a certain kind.

Predicate logic has symbols for properties, symbols to name distinguished objects, and symbols for "for all" and "there exists" 1.

Homework (not actually assigned):

1. Show the following, by giving a proof. Compute the number of lines in the corresponding truth table.
(a) (¬R → ¬S), S |- ¬¬R
(b) (¬O → ¬G), A, (A → G) |- O
(c) (¬L → ¬P), (¬P → ¬W), (¬C → P) |- ¬L → C
(d) (¬D → M), (D → S), Y, (Y → ¬S) |- M
(e) (C → Q), (E → ¬¬Q) |- (¬Q → ¬(C → E))

2. Show the following, by giving a proof. Compute the number of lines in the corresponding truth table.
(a) (S → H), (W → ¬H) |- (S → ¬W)
(b) (F → (E v M)), (E → ¬F) |- (F → M)
(c) ¬(BvS), ¬A, (M → (A & S)) |- ¬M
(d) L, (TC → EC), (L → ¬(TC v EC)), (T → TC), (EC → ¬E) |- (¬E v ¬T)

Lecture 18

Collect homework #11.

Last time, we saw that certain kinds of arguments just cannot be adequately translated into propositional language.

Predicate languages have the following kinds of symbols:

logical connectives: ¬, &, v, →, and parentheses (,) as for propositional languages

Predicate symbols: F, G, H,....

We use 1-place predicate symbols to stand for properties such as being mortal, which apply to a single object. We may use 2-place predicate symbols to stand for relations such as being cousins, which apply to a pair of objects. We may use 3-place predicate symbols for relations such as being between, which apply to a triple of objects. Any finite arity is allowed.
=

individual constants, or names: m, n; a, b, ...

The author has two sets of constants; I prefer just to watch how constants are used.

individual variables: x, y, z, ...

quantifiers: ∃ (there exists) and ( ) (for all)

We are almost ready to define class of well formed formulas for predicate logic. There are some preliminary definitions.

We refer to constants and variables as terms.

The atomic formulas have one of two forms:

(a) t = t', where t and t' are terms
(b) $Ft_1,...,t_n$, where F is an n-place predicate symbol and $t_1,...,t_n$ are terms.

1. An atomic formula is a wff.
2. If A is a wff, then so is ¬A.
3. If A and B are wff's, then so are (A & B), (A v B), (A → B).
4. If A is a wff and v is a variable, then (∃v) A and (v) A are wff's.
5. Nothing is a wff unless it can be obtained by finitely many applications of (1)-(4).

Sample wff's:

1. (x) Fx, saying that all objects of the kind we are considering have property F.

2. (Fm & (∃x) (¬ x = m)), saying that the object named m has property F, and there is another object different from m.

Suppose we change the language and include a 2-place predicate symbol L. Then we have atomic formulas of the form Ltt', where t and t' are terms.

<u>Sample wff's</u>: Lxy, ¬Lmm, (∃x) Lmx

Example 1:

Premises: 1. All men are mortal.
2. Socrates is a man.
Conclusion: Socrates is mortal.

Translate this argument, letting F--property of being a man, G--property of being mortal, m--name for Socrates.

Premises: 1. (x) (Fx → Gx)
2. Fm
Conclusion: Gm

We do not yet have rules of inference. We have only intuitive notion of truth. Using that, does argument look hopeful ?

<u>Example 2</u>:

Premises: 1. Someone lost his or her book.
2. Anyone who lost the book will be unable to read the chapter.
Conclusion: Someone will be unable to read the chapter.

Let L--property of having lost book, K--property of being able to read the chapter

Premises: 1. (∃x) Lx
2. (x) (Lx → ¬Kx)
Conclusion: (∃x) ¬Kx

<u>Example 3</u>: Let L be a 2-place predicate standing for the usual ordering on the natural numbers, and let m be a name for 0. Then Lxy says that x is less than y; x = m says that x is equal to 0.

(a) Write a wff saying that 0 is the first number (less than or equal to any number).

(x) (m = x ∨ Lmx)

(b) What does the following wff say ?

(x) (∃y) Lxy

There is no greatest number; i.e., for each number, there is one which is greater.

Example 4: Letting P stand for the relation (among people) of parent (Pxy says that y is a parent of x), translate the following:

(a) Everyone has a parent.

(x) (∃y) Pxy

(b) Not everyone is a parent.

(∃y) (x) ¬Pxy

(c) Noone is the parent of everyone.

¬(∃y) (x) Pxy, or (y) (∃x) ¬Pxy

Example 5: Let C stand for the 4-place relation (among points in a plane), saying that the line segment from the first to the second has the same length as that from the third to the fourth.
Let m, n, p name the vertices of a triangle.

(a) What does Cmnnp say ?

The triangle is isosceles.

(b) What does (Cmnnp & Cmnmp) say ?

The triangle is equilateral.

Homework #12: p. 103: 1 a-k, q-x

Homework #12: p. 103: 1a-k, q-x

(a) Tabby is a cat. Cm
(b) Rover is not a cat. $\neg Cm$
(c) Some lambs are fleeced. $(\exists x)\ (Lx \ \&\ Fx)$
(d) All lambs are fleeced. $(x)\ (Lx \rightarrow Fx)$
(e) Only lambs are fleeced. $(x)\ (Fx \rightarrow Lx)$
(f) No dog is fleeced. $\neg(\exists x)\ (Dx \ \&\ Fx)$, $(x)\ (Fx \rightarrow \neg Dx)$ or $(x)\ (Dx \rightarrow \neg Fx)$ (all equivalent)
(g) Some dogs aren't fleeced. $(\exists x)\ (Dx \ \&\ \neg Fx)$
(h) Spot is a fleeced dog. $(Dm \ \&\ Fm)$
(i) Brutus killed Caesar. Kmn
(j) Someone killed Caesar. $(\exists x)\ Kxn$
(k) Brutus killed someone. $(\exists y)\ Kmy$
(q) There is a town between London and Stratford. $(\exists x)\ (Tx \ \&\ Bmxn)$
(r) Every woman owns a dog. $(x)\ (Wx \rightarrow \exists y\ (Dy \ \&\ Oxy))$
(s) Some girls like every sport. $(\exists x)\ (Gx \ \&\ (y)\ (Sy \rightarrow Lxy))$
(t) Every intelligent voter casts a ballot. $(x)\ ((Ix \ \&\ Vx) \rightarrow (\exists y)\ (By \ \&\ Cxy))$
(u) Each voter casts an intelligent ballot. $(x)\ (Vx \rightarrow (\exists y)\ ((By \ \&\ Iy) \ \&\ Cxy))$
(v) Tom likes an individualistic sport. Ambiguous.
$(\exists x)\ ((Sx \ \&\ Ix) \ \&\ Lmx)$--Tom likes some individualistic sport.
$(x)\ ((Sx \ \&\ Lmx) \rightarrow Ix)$--Tom only like a sport if it is individualistic.
(w) Some girls like a fast-moving sport. Ambiguous.
$(\exists x)\ (Gx \ \&\ (\exists y)\ ((Sy \ \&\ Fy) \ \&\ Lxy))$--There are girls who like some fast-moving sport.
$(\exists x)\ (Gx \ \&\ (y)\ ((Sy \ \&\ Lxy) \rightarrow Fy))$--There are girls who only like a sport if it is fast-moving.
$(\exists x)\ (Gx \ \&\ (y)((Sy \ \&\ Fy) \rightarrow Lxy)$--There are girls who like all fast-moving sports.
(x) Some boys like only fast-moving sports. Ambiguous.
$(\exists x)\ (Bx \ \&\ (y)\ ((Lxy \ \&\ Sy) \rightarrow Fy))$--There are boys who only like a sport if it is fast-moving. $(\exists x)\ (Bx \ \&\ (y)\ (Lxy \rightarrow (Sy \ \&\ Fy))$--There are boys who don't like food, etc., only sports, and among the sports, only those which are fast-moving.

Lecture 19

Return #11 and collect homework #12.

Recall Exam. II is next Wednesday. Help session Tuesday, Room 214, MCC, 7-9.

Practice exam. is available this time.

Last time, we began predicate logic.

Definition: (1) an atomic formula is a wff ($t_1 = t_2$, or $Ft_1...t_n$)
(2) if A is a wff, so is ¬A
(3) if A, B are wff's, so are (A & B), (A ∨ B), (A → B)
(4) if A is a wff, so are (v) A, (∃v) A
(5) nothing is a wff unless we get it by finitely many applications of (1)-(4).

Example: Show that (x) (∃y) (Fxy → ¬x = y)

As for propositional logic, we give a formation sequence.

1.	Fxy	(1)
2.	x = y	(1)
3.	¬x = y	(2)
4.	(Fxy → ¬x = y)	(3)
5.	(∃y) (Fxy → ¬x = y)	(4)
6.	(x) (∃y) (Fxy → ¬x = y)	(5)

Further examples to translate.

Example 1: Let L be a 2-place predicate standing for the usual ordering on the natural numbers, and let m be a name for 0. Then Lxy says that x is less than y; x = m says that x is equal to 0.

(a) Write a wff saying that 0 is the first number (i.e., for all x, $0 \leq x$).

(x) (m = x ∨ Lmx)

(b) What does the following wff say ?

(x) (∃y) Lxy

There is no greatest number; i.e., for each x, there exists y such that $x < y$.

(c) Compare the following.

$(\exists y)\ (x)\ Lxy$

There is some number which is greater than any number (false)

(d) $(x)\ (y)\ (Lxy \rightarrow (\exists z)\ (Lxz\ \&\ Lzy))$

For any pair of numbers, one less than the other, there is some number in between. False in the natural numbers, true of the rationals or the reals.

Example 2: Letting P stand for the relation (among people) of parent (Pxy says that y is a parent of x), translate the following:

(a) Everyone has a parent.

$(x)\ (\exists y)\ Pxy$

(b) Not everyone is a parent.

$(\exists y)\ (x)\ \neg Pxy$

(c) No-one is the parent of everyone.

$\neg(\exists y)\ (x)\ Pxy$, or $(y)\ (\exists x)\ \neg Pxy$

(d) Everyone has exactly two parents.

$(x)\ (\exists y)\ (\exists z)\ (\neg\ y = z\ \&\ (u)\ (Pxu \leftrightarrow u = y \vee u = z)$

Example 3: Let C stand for the 4-place relation (among points in a plane), saying that the line segment from the first to the second has the same length as that from the third to the fourth.
Let m, n, p name the vertices of a triangle.

(a) What does Cmnnp say ?

The triangle is isosceles.

(b) What does (Cmnnp & Cmnmp) say ?

The triangle is equilateral.

(c) Let m and n name two points. Cound you write a formula true of just those points on the circle with center m and passing through point n ?

Cmxmn

Homework #13 on handout (more translations).

Solutions to #13

I. (a)--(f)'
(b)--(i)'
(c)--(c)'
(d)--(m)'
(e)--(b)'
(f)--(g)'
(g)--(j)'
(h)--(l)'
(i)--(h)'
(j)--(a)'
(k)--(k)'
(l)--(d)'
(m)--(q)'
(n)--(t)'
(o)--(r)'
(p)--(o)'
(q)--(e)'
(r)--(p)'
(s)--(n)'
(t)--(s)'

II. (a) P7--7 is a prime
(b) (E2 & P2)--2 is an even prime
(c) (x) (D2x → Ex)--all numbers which are divisible by 2 are even
(d) (∃x) (Ex & Ex6)--there is an even number which divides 6
(e) (x) (¬Ex → D2x)--any number which is not even is not divisible by 2
(f) (x) (Ex & (y) (Dxy → Ey))--for any number which is even, the numbers which it divides are all even
(g) (x) (Px → (∃y) (Ey & Dxy))--for any prime number, there is an even number which it divides, or each prime number divides some even number
(h) (x) (Ox → (y) (Py → Dxy)--for any odd number and any prime, the odd number does not divide the prime, or no odd number divides a prime
(i) (∃x) (Ex & Px) & ¬(∃x)((Ex & Px) & (∃y) (¬x = y & Ey & Py))--there is an even prime, but there does not exist an even prime with another even prime distinct from it, or there exists one and only one even prime

Lecture 20

Reminder about Exam II Wednesday, help session Tuesday 7-9

Discuss homework #13.

Classify wff's. Some are sentences, some not.

Rougly speaking, a sentence is a formula in which all occurrences of variables are introduced by quantifiers. When a variable occurs without being introduced by a quantifier it is said to occur free. So, a sentence is a formula in which no variables occur free.

Examples: Lmn, (∃x) Pxn, (x) Lxx, (x) (y) Lxy are sentences, while Lxy, (x) Lxy, (Fx & (x) Gx) are not sentences.

In Lxy, x and y both occur free. In (x) Lxy, only y occurs free. In (Fx & (x) Gx), x occurs free (it also occurs later with a quantifier).

Formulas with free variables assert something about those variables.

Suppose we want to say y and z are parents of x, using Puv to say v is parent of u.

We may write (Pxy & Pxz). We are saying something about the free variables x, y, and z.

Suppose we want to say x has two distinct parents, we may write (∃y) (∃z) (Pxy & Pxz & ¬ y = z). We are saying something about the free variable x.

Suppose we want to say y and z are parents of someone (co-parents).

We may write ∃x (Pxy & Pxz). We are saying something about the free variables y and z.

If a variable is introduced by a quantifier, we could change the variable, within moderation, without changing what the formula says.

So, for example, to say that Nancy is a parent, using Pxy for y is a parent of x, and n for Nancy, we may write (∃x) Pxn, or (∃y) Pyn. The two are equivalent, by our intuitive notion of truth.

To say that everyone has a parent, we may write (x) (∃y) Pxy or (z) (∃y) Pzy. The two are equivalent.

What if we write (y) (∃y) Pyy. This is not equivalent to the sentences above. It says that everyone is his or her own parent.

To say that x has played forward and everyone has played goalie, using F for forward and G for goalie, we may write (Fx & (x) Gx), or, more naturally, (Fx & (y) Gy). The two are equivalent.

To say that there is some number less than y, using Lxy for x is less than y, we may write (∃x) Lxy, or (∃z) Lzy. The two are equivalent.

What if we write $(\exists y)$ Lyy ? This is not equivalent to the formulas above--it says that some number is less than itself.

Solutions to #14 (and corrections):

A. Show, by exhibiting a formation sequence, that the following are formulas of predicate logic. For each one, say whether it is a sentence, and if not, give the free variables. Here G is a 1-place predicate, F is a 2-place predicate, and m and n are constants.

1. Gx, Fxy, ¬Fxy, (Gx & ¬Fxy), (∃y) (Gx & ¬Fxy), (x) (∃y) (Gx & ¬Fxy); sentence
2. Gx, Fxy, (∃y) Fxy, (Gx & (∃y) Fxy), ((Gx & (∃y) Fxy) → Gx); not a sentence, x occurs free
3. Gm, Gn, (Gm & Gn); sentence
4. Fxy, Fxz, Fzy, (Fxz & Fzy), (∃z) (Fxz & Fzy), (Fxy → (∃z) (Fxz & Fzy)); not a sentence x,y occur free

B. Letting B be a 2-place predicate saying that the second is a brother of the first, write formulas saying the following:

1. y and z are both brothers of x; (Bxy & Bxz)
2. x has two distinct brothers; (∃y) (∃z) ((Bxy & Bxz) & ¬y = z)
3. x and y each have a brother; ((∃u) Bxu & (∃v) Byv)
4. x and y have a brother in common (i.e., someone is a brother of both x and y); (∃z) (Bxz & Byz)

C. Re-write your formulas 2, 3, and 4 from part B, changing the variables which are introduced by quantifiers, without changing what the formulas say.

2. (∃u) (∃v) ((Bxu & Bxv) & ¬u = v)
3. ((∃z) Bxu & (∃z) Byz)
4. (∃u) (Bxu & Byu)

Solutions to Practice Exam. II

1. (a) P Q (P → Q) ¬(P → Q) (P ∨ Q) (¬(P → Q) & (P ∨ Q)); contingent

P	Q	(P → Q)	¬(P → Q)	(P ∨ Q)	(¬(P → Q) & (P ∨ Q))
F	F	T	F	F	F
F	T	T	F	T	F
T	F	F	T	T	T
T	T	T	F	T	F

(b) P Q R (R → P) (Q → (R → P)) (P → (Q → (R → P))); tautologous

P	Q	R	(R → P)	(Q → (R → P))	(P → (Q → (R → P)))
F	F	F	T	T	T
F	F	T	F	T	T
F	T	F	T	T	T
F	T	T	F	F	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

2. (a) P Q R (P → ¬Q) (Q ∨ R) (¬R → P);

	P	Q	R	(P → ¬Q)	(Q ∨ R)	(¬R → P)
1.	F	F	F	T	F	F
2.	F	F	T	T	T	T
3.	F	T	F	T	T	F
4.	F	T	T	T	T	T
5.	T	F	F	T	F	T
6.	T	F	T	T	T	T
7.	T	T	F	F	T	T
8.	T	T	T	F	T	T

(b) (P → ¬Q), (Q ∨ R) true on lines 2, 3, 4, 6;

(c) (P → ¬Q), (Q ∨ R) ⊭ (¬R ∅ P) , witnessed by line 3

3. φ, ¬ψ |- ¬(φ → ψ)

1	1. φ	A	1,3	4. ψ	1,3	MPP
2	2. ¬ψ	A	1,2,3	5. (ψ & ¬ψ)	4,2	&I
3	3. (φ → ψ)	A	1,2	6. ¬(φ → ψ)	3,5	RAA

4. (a) If |= φ, then |= ((φ & ψ) ↔ ψ), true. (b) If |= φ and φ |= ψ, then |= ψ, true.
(c) If φ |= ψ and ψ |= θ, then φ |= θ, true. (d) If (φ ∨ ψ) |= θ, then φ |= θ or ψ |= θ, true.

5. Letting P--A |= φ and Q--A |- φ, (a) Soundness is (Q → P); (b) Completeness is (P → Q).

6. Letting F--flat tire, T--on time, S--speeding, the premises are (¬F → (S → T)), (¬S → ¬T), and the conclusion is ¬T.

7. (Q → (R → S)), (R → (Q → ¬S)) ⊭ ¬(R & Q); let one of R, Q be false (in particular let Q, R, S all be false). Then the premises are true and the conclusion is false.

8. (P → R), (Q → (R → S)), (¬Q → P) ⊭ (P ∅ S) ; let P be true, S false, R true, Q false. Then the premises are true and the conclusion is false.

9. ((P ∨ Q) → R), (Q → (R → S)) ⊭ (S ⁄ ¬P) ; let S be false, P true, R true, Q false.

10. (P ∨ (Q & R)), (P → R), ¬Q |- (P & R). If the premises hold, Q is false, P is true, and R is true, so the conclusion holds.

Lecture 21

Review, problem session

Meeting 22

Exam. II

Lecture 23

Hand back exams, with homework scores.

Scale on exam. II

90↑ A
80↑ B
65↑ C
50↑ D

On handout from first day of class, I indicated that homework and in class exams each account for 15% of grade. So far, we have had two in class exams and 14 homework assignments. In calculating midsemester grades, I totaled scores on exams (each worth 100 points) and percentage of homework turned in.

Ask about groups.

Papers. Give further references.

Dawson, J., Logical Dilemmas: The Life and Work of Kurt Gö del (book is now out).

Quine, ed., The Ways of Paradox (includes Quine's article from Scientific American, plus papers by other authors)

Encyclopedia of Philosophy, vol. 8, under heading "Paradox"

What can a machine do ?

Recognize that something is a wff of propositional logic, or of predicate logic.

Can you tell just reading from left to right ?

Solutions to Exam. II

1. Formula (¬(P → Q) → P) is tautologous.

P	Q	(P → Q)	¬(P → Q)	(¬(P → Q) → P)
F	F	T	F	T
F	T	T	F	T
T	F	F	T	T
T	T	T	F	T

2. Formula ((P ∨ (Q & R)) & ¬(P ∨ R)) is inconsistent.

P	Q	R	(Q & R)	(P ∨ (Q & R))	(P ∨ R)	¬(P ∨ R)	((P ∨ (Q & R)) & ¬(P ∨ R))
F	F	F	F	F	F	T	F
F	F	T	F	F	T	F	F
F	T	F	F	F	F	T	F
T	T	T	T	T	T	F	F
T	F	F	F	T	T	F	F
T	F	T	F	T	T	F	F
T	T	F	T	T	T	F	F
T	T	T	T	T	T	F	F

3. It is not the case that (P → Q), (Q → ¬P) ⊨ ¬Q. If P and Q are both false, the premises are true and the conclusion is false.

4. It is true that (P → ((Q ∨ R) → S)), (P → ((¬Q ∨ ¬R) → ¬S), (Q & ¬R) ⊨ ¬P. If the conclusion fails, P is true. If the 3rd premise is true, Q is true and R false. Then if the 1st premise is true, S is true, while if the 2nd premise is true, S is false. Any assignment of truth values which makes the premises all true makes the conclusion true.

5. <u>Completeness Theorem</u>: For any propositional statements $A_1,\ldots,A_n$ and B, if B is logically implied by $A_1,\ldots,A_n$, then B is provable from $A_1,\ldots,A_n$; i.e., if $A_1,\ldots,A_n \models B$, then $A_1,\ldots,A_n \vdash B$.

6. (H & H') ∨ (E & E') ∨ (W & W'), ¬W', ¬E ⊨ (H & H')

7. (a)--(ii), (b)--(i), (c)--(iv), (d)--(iii)

8. (a) Emm--a sentence; (b) (∃y) (Exy & ¬x = y)--not a sentence, x occurs free

9. (x) (y) (¬x = y → (∃z) Bxzy)

10. Pxy, (∃y) Pxy, (x) (∃y) Pxy

Sign-up sheet for people not currently participating in study group and wishing to join

Name	Phone number (optional)

Lecture 24 (after Spring Break)

Scale on exam. II

90↑ A
80↑ B
65↑ C
50↑ D

Hand back unclaimed exams., solutions.

On handout from first day of class, I indicated that homework and in class exams each account for 15% of grade. So far, we have had two in class exams and 14 homework assignments. In calculating midsemester grades, I totaled scores on exams (each worth 100 points) and percentage on homework.

Ask about groups.

Papers. Give further references.

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Quine, ed., The Ways of Paradox (includes Quine's article from Scientific American, plus papers by other authors)

Encyclopedia of Philosophy, vol. 8, under heading "Paradox"
Explain calculation of mid-semester grade.

Hand out copies of sample paper.

So far, we have done only a little with predicate logic, translating, defining wff's, classifying occurrences of variables as free or not, and classifying wff's as sentences or not.

Notation: We write A(x) to indicate that A is a formula with at most x occurring free. Then A(c) be the sentence which results from substituting the constant c for the free occurrences of x.
Similarly, for other variables, constants.

Example: If A(x) is the formula (Fx → Gx), then A(m) is the sentence (Fm → Gm)

We are about to begin on proofs in predicate logic. The rules will apply to sentences. We will use notation $A_1,...,A_n$ |- B to mean that there is a proof of B from $A_1,...,A_n$.

We will use all of the old rules, from propositional logic. There are some new rules, for quantifiers, equality.

Here is the first new rule.

<u>Universal Elimination</u> (UE)

From a sentence of the form (v) A(v), where v is a variable, we may conclude A(c) for any constant c.
The line referred to is that of (v) A(v), and the assumptions are those behind (v) A(v).

<u>Example 1</u>: Show that (x) (Fx → G), Fm |- Gm.

This is the famous syllogism, "All men are mortal ...."

1	1. (x) (Fx → Gx)		A
2	2. Fm		A
1	3. (Fm → Gm)	1	UE
1,2	4. Gm	2,3	MPP

<u>Example 2</u>: Show that (x) (y) (Fxy → Fyx), Fmn |- Fnm

1	1. (x) (y) (Fxy → Fyx)		A
2	2. Fmn		A
1	3. (y) (Fmy → Fym)	1	UE
1	4. (Fmn → Fnm)	3	UE
1,2	5. Fnm	2,4	MPP

<u>Example 3</u>: Show that ¬Gm |- ¬(x) Gx

1	1. ¬Gm		A
2	2. (x) Gx		A (aim for contradiction and use RAA)
2	3. Gm	2	UE
1,2	4. (Gm & ¬Gm)	3,1	&I
1	5. ¬(x) Gx	2,4	RAA

Here is the second new rule.

Universal Introduction (UI)

To prove (v) A(v), it is enough to prove A(c), where c is an <u>arbitrary</u> constant; i.e., one not appearing in any of the assumptions that we use for A(c). The line referred to is the one where we have A(c), and the assumptions are those behind A(c).

The rule is natural. Thinking of natural numbers, when you want to argue that (x) A(x) is true, what you do is prove A(n) for an <u>arbitrary</u> number n. It would not work to prove A(0), 0 is a special number, mentioned in the basic assumptions you might use. Similarly, when you want to argue that (x) A(x) is true about people, what you do is prove A(c) for an <u>arbitrary</u> person c.

<u>Example 1</u>: Show that (x) Fx, (x) (Fx → Gx) |- (x) Gx

1	1. (x) Fx		A
2	2. (x) (Fx → Gx)		A
1	3. Fc	1	UE
2	4. (Fc → Gc)	2	UE
1,2	5. Gc	3,4	MPP
1,2	6. (x) Gx	5	UI

<u>Example 2</u>: Is the following a good proof ?

1	1. Em	A	
1	2. (x) Ex	1	UI

Line 1 is itself an assumption in whch m appears, so m is not an arbitrary constant. We are not justified in applying UI.

Having proved A(c), where c is a constant not mentioned in the assumptions, we can only conclude (x) A(x) if A(c) is really the result of replacing all free occurrences of x in A(x) by the constant c.

<u>Example 3</u>: Is the following a good proof ?

1	1. (∃x) ¬x = c	A	
1	2. (x) (∃x) ¬x = x	1	UI

In (∃x) ¬x = x, x is not free. When we replace the free occurrences of x by c, nothing happens. We certainly don't get (∃x) ¬x = c. We are not justified in applying UI.

We will say more about how to recognize situations like this later.

<u>Example 3</u>: Show that (x) (y) Gxy |- (x) Gxx

1. (x) (y) Gxy
2. (y) Gcy
3. Gcc
4. (x) Gxx

<u>Example 4</u>: Show that (x) (Fx ∨ Gx), (x) ¬Fx |- (x) Gx

1	1. (x) (Fx ∨ Gx)			A
2	2. (x) ¬Fx			A
1	3. (Fc ∨ Gc)	1		UE (try to prove Gc, using ∨E)
4	4. Fc			A (try to prove Gc from Fc)
2	5. ¬Fc	2	UE	

6	6. ¬Gc		A (aim for contr., use RAA to get ¬¬Gc)
2,4	7. (Fc & ¬Fc)	4,5	&I
2,4	8. ¬¬Gc	6,7	RAA
2,4	9. Gc	8	DN
10	10. Gc		A (prove Gc from Gc)
1,2	11. Gc	3,4,9,10,10	∨E
1,2	12. (x) Gx	11	UI

Homework #15: p. 110: 1,2(i)a,b,c

Homework #7: in text, p. 41: 1d,e,f,g,h,i,j (these were very difficult--feel good if you can see how to do parts of some problems)

Homework #14:

A. Show, by exhibiting a formation sequence, that the following are formulas of predicate logic. For each one, say whether it is a sentence, and if not, give the free variables. Here G is a 1-place predicate, F is a 2-place predicate, and m and n are constants.

1. $(x)\ (\exists y)\ (Gx\ \&\ \neg Fxy)$
2. $((Gx\ \&\ (\exists y)\ Fxy) \rightarrow Gx)$
3. $(Gm\ \&\ Gn)$; sentence
4. $(Fxy \rightarrow (\exists z)\ (Fxz\ \&\ Fzy))$

B. Letting B be a 2-place predicate saying that the second is a brother of the first, write formulas saying the following:

1. y and z are both brothers of x
2. x has two distinct brothers
3. x and y each have a brother
4. x and y have a brother in common

C. Re-write your formulas 2, 3, and 4 from part B, changing the variables which are introduced by quantifiers, without changing what the formulas say.

Homework #15: p. 110: 1, 2(i)a,b,c

1. (a) Dm, (x) (Dx → Bx) |- Bm

1	1. Dm		A
2	2. (x) (Dx → Bx)		A
2	3. (Dm → Bm)	2	UE
1,2	4. Bm	3,1	MPP

(b) Bm, (x) (Dx → ¬Bx) |- ¬Dm

1	1. Bm		A
2	2. (x) (Dx → ¬Bx)		A
2	3. (Dm → ¬Bm)	2	UE
1	4. ¬¬Bm	1	DN
1,2	5. ¬Dm	3,4	MTT

(c) ¬Dm, (x) (Hx → Dx) |- ¬Hm

1	1. ¬Dm		A
2	2. (x) (Hx → Dx)		A
2	3. (Hm → Dm)	2	UE
1,2	4. ¬Hm	3,1	MTT

(d) (x) ((Mx & Nx) → Sx), ¬Sn, Mn |- ¬Nn

1	1. (x) ((Mx & Nx) → Sx)		A
2	2. ¬Sn		A
3	3. Mn		A
1	4. ((Mn & Nn) → Sn)	1	UE
1,2	5. ¬(Mn & Nn)	4	MTT
6	6. Nn	A	
3,6	7. (Mn & Nn)	3,6	&I
1,2,3,6	8. ((Mn & Nn) & ¬(Mn & Nn))	7,5	&I
1,2,3	9. ¬Nn	6,8	RAA

(e) (x) ((Fx & ¬Px) → Kx), Fm, ¬Km |- Pm

1	1. (x) ((Fx & ¬Px) → Kx)		A
2	2. Fm		A
3	3. ¬Km		A
1	4. ((Fm & ¬Pm) → Km)	1	UE
5	5. ¬Pm		A (use RAA)
2,5	6. (Fm & ¬Pm)	2,5	&I
1,2,5	7. Km	4,6	MPP
1,2,3,5	8. (Km & ¬Km)	7,3	&I
1,2,3	9. ¬¬Pm	5,8	RAA
1,2,3	10. Pm	9	DN

2. (i) (a) (x) (Fx → Gx), (x) (Gx → ¬Hx) |- (x) (Fx → ¬Hx)

1	1.	(x) (Fx → Gx)		A
2	2.	(x) (Gx → ¬Hx)		A (prove (Fc → ¬Hc and use UI)
3	3.	Fc		A (prove ¬Hc and use CP)
1	4.	(Fc → Gc)	1	UE
2	5.	(Gc → ¬Hc)	2	UE
1,3	6.	Gc	4,3	MPP
1,2,3	7.	¬Hc	5,6	MPP
1,2	8.	(Fc → ¬Hc)	3,7	CP
1,2	9.	(x) (Fx → ¬Hx)	8	UI

(b) (x) (Fx → ¬Gx), (x) (Hx → Gx) |- (x) (Fx → ¬Hx)

1	1.	(x) (Fx → ¬Gx)		A
2	2.	(x) (Hx → Gx)		A (prove (Fc → ¬Hc) and use UI)
3	3.	Fc		A (prove ¬Hc and use CP)
1	4.	(Fc → ¬Gc)	1	UE
1,3	5.	¬Gc	3,4	MPP
2	6.	(Hc → Gc)	2	UE
1,2,3	7.	¬Hc	6,7	MTT
1,2	8.	(Fc → ¬Hc)	3,7	CP
1,2	9.	(x) (Fx → ¬Hx)	8	UI

(c) (x) (Fx → Gx), (x) (Hx → ¬Gx) |- (x) (Fx → ¬Hx)

1	1.	(x) (Fx → Gx)		A
2	2.	(x) (Hx → ¬Gx)		A (prove (Fc → ¬Hc) and use UI)
3	3.	Fc		A (prove ¬Hc and use CP)
1	4.	(Fc → Gc)	1	UE
2	5.	(Hc → ¬Gc)	2	UE
1,3	6.	Gc	4,3	MPP
1,3	7.	¬¬Gc	6	DN
1,2,3	8.	¬Hc	5,7	MTT
1,2	9.	(Fc → ¬Hc)	3,8	CP
1,2	10.	(x) (Fx → ¬Hx)	9	UI

Lecture 25

Questions on homework #15 ?

Further examples of proofs with old and new rules

Example 1: Show that (x) (Fx → Gx) |- ((x) Fx → (x) Gx)

1	1. (x) (Fx → Gx)	A	
2	2. (x) Fx	A (prove (x) Gx and use CP; prove Gc and use UI)	
1	3. (Fc → Gc) 1	UE	
2	4. Fc	2	UE
1,2	5. Gc	3,4	MPP
1,2	6. (x) Gx	5	UI
1	7. ((x) Fx → (x) Gx) 2,6		CP

Example 2: Show that (x) (y) Gxy |- (x) Gxx

1	1. (x) (y) Gxy	A	
1	2. (y) Gcy	1	UE
1	3. Gcc 2	UE	
1	4. (x) Gxx	3	UI

Example 3: Show that (x) (y) Gxy |- (y) (x) Gxy

1	1. (x) (y) Gxy	A	
1	2. (y) Gcy	1	UE
1	3. Gcd 2	UE	
1	4. (x) Gxd	3	UI
1	5. (y) (x) Gxy 4	UI	

Example 4: Show that ((x) Fx ∨ (x) Gx) |- (x) (Fx ∨ Gx)

1	1. ((x) Fx ∨ (x) Gx)		A (prove (Fc ∨ Gc) and use UI; use ∨E)
2	2. (x) Fx		A
2	3. Fc	2	UE
2	4. (Fc ∨ Gc)	3	∨I
6	6. (x) Gx		A
6	7. Gc	6	UE
6	8. (Fc ∨ Gc)	7	∨I
1	9. (Fc ∨ Gc)	1,2,4,6,8	∨E
1	10. (x) (Fx ∨ Gx)	8	UI

Example 5: Show that (x) (Fx ∨ Gx), (x) ¬Fx |- (x) Gx

1	1. (x) (Fx ∨ Gx)		A
2	2. (x) ¬Fx		A
1	3. (Fc ∨ Gc)	1	UE (try to prove Gc, using ∨E)
4	4. Fc		A (try to prove Gc from Fc)
2	5. ¬Fc	2	UE
6	6. ¬Gc		A (aim for contr., use RAA to get ¬¬Gc)
2,4	7. (Fc & ¬Fc)	4,5	&I
2,4	8. ¬¬Gc	6,7	RAA
2,4	9. Gc	8	DN
10	10. Gc		A (prove Gc from Gc)
1,2	11. Gc	3,4,9,10,10	∨E
1,2	12. (x) Gx	11	UI

Homework #16: p. 110: 2(i) d,e,f, (ii)

Homework #16: p. 110: 2(i)d,e,f, (ii)

(i) (d) (x) (Gx → ¬Fx), (x) (Hx → Gx) |- (x) (Fx → ¬Hx)

1	1. (x) (Gx → ¬Fx)			A
2	2. (x) (Hx → Gx)			A
3	3. Fc			A (prove ¬Hc and use CP)
1	4. (Gc → ¬Fc)		1	UE
2	5. (Hc → Gc)	2	UE	
3	6. ¬¬Fc		3	DN
1,3	7. ¬Gc	4,6	MTT	
1,2,3	8. ¬Hc	5,7	MTT	
1,2	9. (Fc → ¬Hc)		3,8	CP
1,2	10. (x) (Fx → ¬Hx)		9	UI

(e) (x) (Fx → Gx) |- ((x) Fx → (x) Gx)

1	1. (x) (Fx → Gx)			A
2	2. (x) Fx			A (prove (x) Gx and use CP)
1	3. (Fc → Gc)	1	UE	
2	4. Fc		2	UE
1,2	5. Gc		3,4	MPP
1,2	6. (x) Gx		5	UI
1	7. ((x) Fx → (x) Gx)		2,6	CP

(f) (x) ((Fx ∨ Gx) → Hx), (x) ¬Hx |- (x) ¬Fx

1	1. (x) ((Fx ∨ Gx) → Hx)		A	
2	2. (x) ¬Hx		A	
1	3. ((Fc ∨ Gc) → Hc)		1	UE
2	4. ¬Hc	2	UE	
5	5. Fc			A (use RAA)
5	6. (Fc ∨ Gc)		5	∨I
1,5	7. Hc		3,6	MPP
1,2,5	8. (Hc & ¬Hc)		7,4	&I
1,2	9. ¬Fc	5,8	RAA	
1,2	10. (x) ¬Fx		9	UI

(ii) ii a translates into i d
ii b translates into i b
ii c translates into i c
ii d translates into i a

Lecture 26

Questions on #16 ?

So far, our set of rules of proof for predicate logic includes all the propositional rules, plus the two rules for universal quantifiers UE and UI. Here is one more example using these rules, also using ∨E.

Example: Show that ((x) Fx ∨ (x) Gx) |- (x) (Fx ∨ Gx)

1	1. ((x) Fx ∨ (x) Gx)		A (prove (Fc ∨ Gc) and use UI; use ∨E)
2	2. (x) Fx		A
2	3. Fc	2	UE
2	4. (Fc ∨ Gc)	3	∨I
6	6. (x) Gx		A
6	7. Gc	6	UE
6	8. (Fc ∨ Gc)	7	∨I
1	9. (Fc ∨ Gc)	1,2,4,6,8	∨E
1	10. (x) (Fx ∨ Gx)	8	UI

Next, we add two rules for existential quantifiers.

Existential Introduction (EI)

This rule says that from A(c), we get (∃v) A(v), where A(v) is a formula with at most v occurring free, and A(c) is the result of substituting c for the free occurrences of x in this formula. The line referred to is the one where we have A(c), and the assumptions are those behind A(c).

Example 1: Show that (x) Fx |- (∃x) Fx

1	1. (x) Fx		A
1	2. Fc	1	UE
1	3. (∃x) Fx	2	EI

We need to be a little careful in applying EI.

Example 2: Is the following a correct proof ?

1	1. (x) (Fx → ¬c = x)	A	
1	2. (∃x) (x) (Fx → ¬x = x)	1	EI

Line 2 is not justified. There are no free occurrences of x in A(x) = (x) (Fx → ¬x = x). If we substitute c for the free occurrences of x, we do not get (x) (Fx → ¬c = x).

Existential Elimination (EE)

This rule resembles (∨E).

To prove B using (∃x) A(x), we prove B assuming A(c), where c is an arbitrary constant, not in B or A(x), or any other assumption used in getting B. The lines referred to are the one where we have (∃x) A(x), the one where we assume A(c), and the one where we arrive at B from A(c). The assumptions are those used in getting B from A(c), except that (∃x) A(x) replaces A(c).

The rule is natural. Knowing that (∃x) A(x), we say "let c be an x satisfying A(x)", we prove B from that, and we conclude that B follows from (∃x) A(x).

Example 1: Show that (∃x) Fx, (x) (Fx → Gx) |- (∃x) Gx.

1	1. (∃x) Fx		A
2	2. (x) (Fx → Gx)		A
3	3. Fc		A (use EE)
2	4. (Fc → Gc) 2	UE	
2,3	5. Gc	4,3	MPP
2,3	6. (∃x) Gx	5	EI
1,2	7. (∃x) Gx	1,3,6	EE

Lecture 27

Last time, we introduced rules for existential quantifiers.

EI: From A(c), we get (∃x) A(x). Here c need not be an arbitrary constant. We need to be sure that A(c) is result of substituting c for free occurrences of x in A(x). Line referred to is that of A(c), and assumptions are those behind A(c).

EE: To prove B from (∃x) A(x), it is enough to prove B from A(c), where c is an arbitrary constant. Lines referred to are one where we have (∃x) A(x), one where we assume A(c), and one where we obtain B from A(c). Assumptions are those in this proof of B except that A(c) is replaced by (∃x) A(x). To count as "arbitrary" here, c must not appear in B, A(x), or the other assumptions behind B.

<u>Example 1</u>: Show that ¬Oc, (∃x) Ox, (x) (y) (x = y → (Ox → Oy)) |- (∃x) ¬x = c

1	1. ¬Oc		A
2	2. (∃x) Ox		A
3	3. (x) (y) (x = y → (Ox → Oy))		A
4	4. Oa		A (use EE; i.e., prove (∃x) ¬x = c from Oa & then get it from (∃x) ¬x = c; to do this, prove ¬a = c)
5	5. a = c		A (use RAA)
3	6. (y) (a = y → (Oa → Oy))	3	UE
3	7. (a = c → (Oa → Oc))	6	UE
3,5	8. (Oa → Oc)	7,5	MPP
3,4,5	9. Oc	8,4	MPP
1,3,4,5	10. (Oc & ¬Oc)	9,1	&I
1,3,4	11. ¬a = c	5,10	RAA
1,3,4	12. (∃x) ¬x = c	11	EI
1,2,3	13. (∃x) ¬x = c	2,4,12	EE

Try problems in text, p. 116.

1(a) (x) (Fx → Gx), (∃x) ¬Gx |- (∃x) ¬Fx

1	1. (x) (Fx → Gx)		A
2	2. (∃x) ¬Gx		A
3	3. ¬Ga		A (use EE)
1	4. (Fa → Ga)	1	UE
1,3	5. ¬Fa	4,3	MTT
1,3	6. (∃x) ¬Fx	5	EI
1,2	7. (∃x) ¬Fx	2,3,6	EE

(b) (x) (Fx → (Gx & Hx)), (∃x) Fx |- (∃x) Hx

1	1. (x) (Fx → (Gx & Hx))		A

2	2. (∃x) Fx		A	
3	3. Fa		A (use EE)	
1	4. (Fa → (Ga & Ha))	1	UE	
1,3	5. (Ga & Ha)	4,3	MPP	
1,3	6. Ha	5	&E	
1,3	7. (∃x) Hx	6	EI	
1,2	8. (∃x) Hx	2,3,7	EE	

2 (i) (a) (x) (Gx → ¬Hx), (∃x) (Fx & Gx) |- (∃x) (Fx & ¬Hx)

1	1. (x) (Gx → ¬Hx)		A
2	2. (∃x) (Fx & Gx)		A
3	3. (Fa & Ga)		A (use EE)
1	4. (Ga → ¬Ha)	1	UE
3	5. Fa	3	&E
3	6. Ga	3	&E
1,3	7. ¬Ha	4,6	MPP
1,3	8. (Fa & ¬Ha)	5,7	&I
1,3	9. (∃x) (Fx & ¬Hx)	8	EI
1,2	10. (∃x) (Fx & ¬Hx)	2,3,9	EE

Lecture 28

Collect papers.

Further problems on p. 117, 128

2(i) (c) (x) (Hx → ¬Gx), (∃x) (Fx & Gx) |- (∃x) (Fx & ¬Hx)

1	1. (x) (Hx → ¬Gx)		A
2	2. (∃x) (Fx & Gx)		A
3	3. (Fa & Ga)		A
1	4. (Ga → ¬Ga)	1	UE
3	5. Fa	3	&E
3	6. Ga	3	&E
3	7. ¬¬Ga	6	DN
1,3	8. ¬Ha	4,7	MTT
1,3	9. (Fa & ¬Ha)	5,8	&I
1,3	10. (∃x) (Fx & ¬Hx)	9	EI
1,2	11. (∃x) (Fx & ¬Hx)	2,3,10	EE

(e) (∃x) (Gx & Hx), (x) (Gx → Fx) |- (∃x) (Fx & Hx)

1	1. (∃x) (Gx & Hx)		A
2	2. (x) (Gx → Fx)		A
3	3. (Ga & Ha)		A (use EE)
3	4. Ga	3	&E
3	5. Ha	3	&E
2	6. (Ga → Fa)	2	UE
2,3	7. Fa	6,5	MPP
2,3	8. (Fa & Ha)	7,5	&I
2,3	9. (∃x) (Fx & Hx)	8	EI
1,2	10. (∃x) (Fx & Hx)	1,2,9	EE

2(ii) (a) No mountains are climbable, some hills are climbable; therefore, some hills are not mountains.

translation: (x) (Mx → ¬Cx), (∃x) (Hx & Cx) |- (∃x) (Hx & ¬Mx)

This matches form of 2i(c) above if we use H instead of M, G instead of C, F instead of H. Assuming that our rules for predicate logic are sound, we have shown that the argument is sound.

(b) Some mountains are climbable; all mountains are hills; therefore some hills are climbable.

translation: (∃x) (Mx & Cx), (x) (Mx → Hx) |- (∃x) (Hx & Cx)

This matches form of some part of 2i.

Try some problems from p. 128 (harder).

1 (a) (x) (Fx → Gx) |- ((x) ¬Gx → (x) ¬Fx)

1	1. (x) (Fx → Gx)			A	
2	2. (x) ¬Gx			A	(use CP; prove ¬Fa, get (x) ¬Fx)
1	3. (Fa → Ga)	1	UE		
2	4. ¬Ga	2	UE		
1,2	5. ¬Fa	3,4	MTT		
1,2	6. (x) ¬Fx		5	UI	
1	7. (x) ¬Gx ∅ (x) ¬Fx	2,6	CP		

(b) (x) (Fx → Gx) |- ((∃x) ¬Gx → (∃x) ¬Fx)

1	1. (x) (Fx → Gx)		A		
2	2. (∃x) ¬Gx		A	(use CP; prove (∃x) ¬Fx)	
3	3. ¬Ga	A	(use EE)		
1	4. (Fa → Ga)	1	UE		
1,3	5. ¬Fa	4,3	MTT		
1,3	6. (∃x) ¬Fx		5	EI	
1,2	7. (∃x) ¬Fx		2,3,6	EE	
1	8. (∃x) ¬Gx → (∃x) ¬Fx		2,7	CP	

Lecture 29

Hand back papers.

What will we do in the rest of the course ? First, more on translating arguments and giving proofs in predicate logic. I'll also define truth and say a little about soundness and completeness for predicate logic.

Exam. III is April 11. This will be on translation and proofs in predicate logic.

Then come your talks, roughly 15 minutes, based on your papers. These will be given in class April 16, 18, **21, 23, 25,** 28. It is important to come to class on those days. Show respect for each other. Listen to the talks, ask questions. The papers are really excellent, full of interesting bits of history, philosophical insight. There are clear explanations of some opaque sections of the book. There are topics beyond what is in the book. There will be no other assignments during this time, just giving your own talk and listening and learning from other students' talks.

I would like to hear the talks in advance, probably during the period April 14-18.

You should prepare abstract (on a single page), giving
1. the title of your talk,
2. your name, and the names of your collaborators (if any),
3. a paragraph or two saying what the talk is about, and
4. a list of the references you used.

Make copies of your abstract for the whole class, or give it to me in advance, and I will make copies. You may want to prepare slides for the talk.

p. 128

1 (d) $(x)\neg Fx \dashv\vdash \neg(\exists x)Fx$

First, show $(x)\neg Fx \vdash \neg(\exists x)Fx$

1	1.	$(x)\neg Fx$		A
2	2.	$(\exists x)Fx$		A (use RAA)
3	3.	Fa		A (use EE)
1	4.	$\neg Fa$	1	UE
1,3	5.	$(Fa \& \neg Fa)$	3,4	&I

(want contradiction from 2; can't use EE because this contradiction involves a)

3	6.	$\neg(x)\neg Fx$	1,5	RAA
2	7.	$\neg(x)\neg Fx$	2,3,6	EE
1,2	8.	$((x)\neg Fx \& \neg(x)\neg Fx)$	1,7	&I
1	9.	$\neg(\exists x)Fx$	2,8	RAA

Next, show ¬(∃x) Fx |- (x) ¬Fx

1	1. ¬(∃x) Fx		A
2	2. Fa		A (use RAA)
2	3. (∃x) Fx	2	EI
1,2	4. ((∃x) Fx & ¬(∃x) Fx)	3,1	&I
1	5. ¬Fa 2,4	RAA	
1	6. (x) ¬Fx	5	UI

(e) (x)

p. 137: 1 (b) (x) (∃y) (z) Fxyz |- (x) (z) (∃y) Fxyz

1	1. (x) (∃y) (z) Fxyz		A	
1	2. (∃y) (z) Fayz	1	UE	
3	3. (z) Fabz		A	(use EE)
3	4. Fabc	3	UE	
3	5. (∃y) Fayc	4	EI	
1	6. (∃y) Fayc	2,3,5	EE	
1	7. (z) (∃y) Fayz	6	UI	
1	8. (x) (z) (∃y) Fxyz	7	UI	

Homework #17: p. 117 1(c), 2i (b),(d),(f),(g), 2ii; p. 128: 1(c)

Solutions to Homework #17: p. 117 1(c), 2i (b),(d),(f),(g), 2ii; p. 128: 1(c)

p. 117: 1(c) (x) ((Fx ∨ Gx) → Hx), (∃x) ¬Hx |- (∃x) ¬Fx

1	1. (x) ((Fx ∨ Gx) → Hx)		A
2	2. (∃x) ¬Hx		A
3	3. ¬Ha		A (use EE)
1	4. ((Fa ∨ Ga) → Ha)	1	UE
5	5. Fa		A (use RAA)
5	6. (Fa ∨ Ga)	5	∨I
1,5	7. Ha	4,6	MPP
1,3,5	8. (Ha & ¬Ha)	7,3	&I
1,3	9. ¬Fa	5,8	RAA
1,3	10. (∃x) ¬Fx	9	EI
1,2	11. (∃x) ¬Fx	2,3,10	EE

2i(b) (x) (Hx → Gx), (∃x) (Fx & ¬Gx) |- (∃x) (Fx &¬Hx)

1	1. (x) (Hx → Gx)		A
2	2. (∃x) (Fx & ¬Gx)		A
3	3. (Fc & ¬Gc)		A (use EE)
1	4. (Hc → Gc)	1	UE
3	5. Fc	3	&E
3	6. ¬Gc	3	&E
1,3	7. ¬Hc	4,6	MTT
1,3	8. (Fc & ¬Hc)	5,7	&I
1,3	9. (∃x) (Fx & ¬Hx)	8	EI
1,2	10. (∃x) (Fx & ¬Hx)	2,3,9	EE

(d) (x) (Gx → Hx), (∃x) (Gx & Fx) |- (∃x) (Fx & Hx)

1	1. (x) (Gx → Hx)		A
2	2. (∃x) (Gx & Fx)		A
3	3. (Gb & Fb)		A (use EE)
1	4. (Gb → Hb)	1	UE
3	5. Gb	3	&E
3	6. Fb	3	&E
1,3	7. Hb	4,5	MPP
1,3	8. (Fb & Hb)	6,7	&I
1,3	9. (∃x) (Fx & Hx)	8	EI
1,2	10. (∃x) (Fx & Hx)	2,3,9	EE

(f) (x) (Gx → ¬Hx), (∃x) (Gx & Fx) |- (∃x) (Fx & ¬Hx)

1	1. (x) (Gx → ¬Hx)		A
2	2. (∃x) (Gx & Fx)		A
3	3. (Ga & Fa)		A (use EE)
1	4. (Ga → ¬Ha)	1	UE
3	5. Ga	3	&E
3	6. Fa	3	&E
1,3	7. ¬Ha	4,5	MPP
1,3	8. (Fa & ¬Ha)	6,7	&I

1,3	9. (∃x) (Fx & ¬Hx)	8	EI	
1,2	10. (∃x) (Fx & ¬Hx)	2,3,9	EE	

(g) (∃x) (Gx & ¬Hx), (x) (Gx → Fx) |- (∃x) (Fx & ¬Hx)

1	1. (∃x) (Gx & ¬Hx)		A	
2	2. (x) (Gx → Fx)		A	
3	3. (Ga & ¬Ha)		A (use EE)	
2	4. (Ga → Fa)	2	UE	
3	5. Ga		3	&E
3	6. ¬Ha	3	&E	
2,3	7. Fa		4,5	MPP
2,3	8. (Fa & ¬Ha)		7,6	&I
2,3	9. (∃x) (Fx & ¬Hx)		8	EI
1,2	10. (∃x) (Fx & ¬Hx)	1,3,9	EE	

2 ii (a) matches i(c)
ii (b) matches i(e)
ii (c) matches i(b)
ii (d) matches i(d)
ii (e) matches i(a)
ii (f) matches i(g)
ii (g) matches i(f)

p. 128: 1(c) (∃x) ¬Fx -||- ¬(x) Fx

First, show (∃x) ¬Fx |- ¬(x) Fx

1	1. (∃x) ¬Fx		A	
2	2. ¬Fa	A	(use EE)	
3	3. (x) Fx		A	(use RAA)
3	4. Fa	3	UE	
2,3	5. (Fa & ¬Fa) 4,2	&I		
2	6. ¬(x) Fx	3,5	RAA	
1	7. ¬(x) Fx	1,2,6	EE	

Next, show ¬(x) Fx |- (∃x) ¬Fx

1	1. ¬(x) Fx			A
2	2. ¬(∃x) ¬Fx		A	(use RAA)
3	3. ¬Fa		A	(use RAA)
3	4. (∃x) ¬Fx		3	EI
2,3	5. ((∃x) ¬Fx & ¬(∃x) ¬Fx)	4,2	&I	
2	6. Fa		3,5	RAA
2	7. (x) Fx		6	UI
1,2	8. ((x) Fx & ¬(x) Fx)	7,1	&I	
1	9. (∃x) ¬Fx		2,8	RAA

Reminders

Exam. III is Friday, April 11, help session Thursday, April 10, 7-9, Room 214 MCC.

Talks are April 16, 18, 21, 23, 25, 28 (with rehearsals starting April 14). You are obliged to attend all talks.

Lecture 30

Collect homework and hand out solutions.

Further example on p. 128, 2(i).
All women are mortal is interderivable with there are not women who are not mortal.
(x) (Wx → Mx) -||- ¬(∃x) (Wx & ¬Mx)

First, show (x) (Wx → Mx) |- ¬(∃x) (Wx & ¬Mx)

1	1.	(x) (Wx → Mx)	A	
2	2.	(∃x) (Wx & ¬Mx)	A (use RAA)	
3	3.	(Wa & ¬Ma)	A (use EE)	
1	4.	(Wa → Ma)	1	UE
3	5.	Wa	3	&E
1,3	6.	Ma	4,5	MPP
3	7.	¬Ma	3	&E
1,3	8.	(Ma & ¬Ma)	6,7	&I
1,3	9.	¬(∃x) (Wx & ¬Mx)	2,8	RAA (cannot discharge 2)
1,2	10.	¬(∃x) (Wx & ¬Mx)	2,3,9	EE (still have 2)
1,2	11.	(∃x) (Wx & ¬Mx) & ¬(∃x) (Wx & ¬Mx)	2,10	&I
1	12.	¬(∃x) (Wx & ¬Mx)	2,11	RAA

Now, show ¬(∃x) (Wx & ¬Mx) |- (x) (Wx → Mx)

1	1.	¬(∃x) (Wx & ¬Mx)	A	
2	2.	Wa	A (use CP)	
3	3.	¬Ma	A (use RAA)	
2,3	4.	(Wa & ¬Ma)	2,3	&I
2,3	5.	(∃x) (Wx & ¬Mx)	4	EI
1,2,3	6.	(∃x) (Wx & ¬Mx) & ¬(∃x) (Wx & ¬Mx)	5,1	&I
1,2	7.	Ma	3,6	RAA
1	8.	(Wa → Ma)	2,8	CP
1	9.	(x) (Wx → Mx)	8	UI

p. 138: 2 (b) All camels like a gentle driver; some camels do not like Mohammed; Mohammed is a driver; therefore, Mohammed is not gentle.

(x) (y) ((Cx & (Gy & Dy)) → Lxy), (∃x) (Cx & ¬Lxm), Dm |- ¬Gm

1	1.	(x) (y) ((Cx & (Gy & Dy)) → Lxy)	A
2	2.	(∃x) (Cx & ¬Lxm)	A
3	3.	Dm	A
4	4.	(Ca & ¬Lam)	A (use EE)

1	5. (y) ((Ca & (Gy & Dy)) → Lay)	1	UE
1	6. ((Ca & (Gm & Dm)) → Lam)	5	UE
7	7. Gm	A (use RAA)	
4	8. Ca	4	&E
3,7	9. (Gm & Dm)	7,3	&I
3,4,7	10. (Ca & (Gm & Dm))	8,9	&I
1,3,4,7	11. Lam	6,10	MPP
4	12. ¬Lam	4	&E
1,3,4,7	13. (Lam & ¬Lam)	11,12	&I
1,3,4	14. ¬Gm	7,13	RAA
1,2,3	15. ¬Gm	2,4,14	EE

Homework #18:
p. 128 1(d), 2ii;
p. 137 1 (b), (c), 2 (c), (d), (e).

Solutions to Problems on #18

p. 128: 1(d) (x) ¬Fx -||- ¬(∃x) Fx

First, show (x) ¬Fx |- ¬(∃x) Fx

1	1.	(x) ¬Fx		A
2	2.	(∃x) Fx		A (use RAA)
3	3.	Fc		A (use EE)
1	4.	¬Fc	1	UE
1,3	5.	(Fc &¬Fc)	3,4	&I
1,3	6.	¬(∃x) Fx	2,5	RAA
1,2	7.	¬(∃x) Fx	2,4,6	EE
1,2	8.	(∃x) Fx & ¬(∃x) Fx	2,7	&I
1	9.	¬(∃x) Fx	2,8	RAA

Now, show ¬(∃x) Fx |- (x) ¬Fx

1	1.	¬(∃x) Fx		A
2	2.	Fc		A (use RAA)
2	3.	(∃x) Fx	2	EI
1,2	4.	(∃x) Fx & ¬(∃x) Fx	3,1	&I
1	5.	¬Fc	2,4	RAA
1	6.	(x) ¬Fx	5	UI

2ii Show (x) (Fx → ¬Gx) (no men are mortal) is interderivable with ¬(∃x) (Fx & Gx) (there are not men who are mortal). This is 1 (e).

p. 137 (b) (x) (∃y) (z) Fxyz |- (x) (z) (∃y) Fxyz

1	1.	(x) (∃y) (z) Fxyz		A
1	2.	(∃y) (z) Fayz	1	UE
3	3.	(z) Fabz		A (use EE)
3	4.	Fabc	3	UE
3	5.	(∃y) Fayc	4	EI
1	6.	(∃y) Fayc	2,3,5	EE
1	7.	(z) (∃y) Fayz	6	UI
1	8.	(x) (z) (∃y) Fxyz	7	UI

(c) (∃x) (∃y) (z) Fxyz |- (z) (∃y) (∃x) Fxyz

1	1.	(∃x) (∃y) (z) Fxyz		A
2	2.	(∃y) (z) Fayz		A (use EE)
3	3.	(z) Fabz		A (use EE)
3	4.	Fabc	3	UE
3	5.	(∃x) Fxbc	4	EI
3	6.	(∃y) (∃x) Fxyc	5	EI
3	7.	(z) (∃y) (∃x) Fxyz	6	UI
2	8.	(z) (∃y) (∃x) Fxyz	2,3,7	EE
1	9.	(z) (∃y) (∃x) Fxyz	1,2,8	EE

2 (c) All camels are highly strung animals; some drivers like no highly strong animals; therefore, some drivers do not like any camels.

(x) (Cx → Hx), (∃x) (Dx & (y) (Hy → ¬Lxy)) |- (∃x) (Dx & (y) (Cy → ¬Lxy))

1	1. (x) (Cx → Hx)		A
2	2. (∃x) (Dx & (y) (Hy → ¬Lxy))		A
3	3. (Da & (y) (Hy → ¬Lay))		A (use EE)
3	4. Da	3	&E
3	5. (y) (Hy → ¬Lay)	3	&E
3	6. (Hc → ¬Lac)	5	UE
1	7. (Cc → Hc)	1	UE
8	8. Cc		A (use CP)
1,8	9. Hc	7,8	MPP
1,3,8	10. ¬Lac	6,9	MPP
1,3	11. (Cc → ¬Lac)	8,10	CP
1,3	12. (y) (Cy → ¬Lay)	11	UI
1,3	13. (Da & (y) (Cy → ¬Lay))	4,12	&I
1,3	14. (∃x) (Dx & (y) (Cy → ¬Lxy))	13	EI
1,2	15. (∃x) (Dx & (y) (Cy → ¬Lxy))	2,3,14	EE

(d) Some dogs like William; all boys like any dog; William is a boy; therefore, there is something which likes and is liked by William.

(∃x) (Dx & Lxw), (x) (y) ((Bx & Dy) → Lxy), Bw |- (∃x) (Lxw & Lwx)

1	1. (∃x) (Dx & Lxw)		A
2	2. (x) (y) ((Bx & Dy) → Lxy)		A
3	3. Bw		A
4	4. (Da & Law)		A (use EE)
2	5. (y) ((Bw & Dy) → Lwy)	2	UE
2	6. ((Bw & Da) Æ Lwa)	5	UE
4	7. Da	4	&E
4	8. Law	4	&E
3,4	9. (Bw & Da)	3,7	&I
2,3,4	10. Lwa	6,9	MPP
2,3,4	11. (Law & Lwa)	8,10	MPP
2,3,4	12. (∃x) (Lxw & Lwx)	11	EI
1,2,3	13. (∃x) (Lxw & Lwx)	1,4,12	EE

(e) A whale is a mammal; some fish are whales; all fish have tails; therefore, some fishes' tails are mammals' tails.

(x) (Wx → Mx), (∃x) (Fx & Wx), (x) (Fx → (∃y) Txy) |- (∃x) ((∃y) (Fy & Tyx) & (∃y) (My & Tyx))

1	1. (x) (Wx → Mx)		A
2	2. (∃x) (Fx & Wx)		A
3	3. (x) (Fx → (∃y) Txy)		A
4	4. (Fa & Wa)		A (use EE)
1	5. (Wa → Ma)	1	UE
3	6. (Fa → (∃y) Txy)	3	UE
4	7. Fa	4	&E
4	8. Wa	4	&E
1,4	9. Ma	5,8	MPP

3,4	10. (∃y) Tay	6,7	MPP
11	11. Tab		A (use EE)
4,11	12. (Fa & Tab)	7,11	&I
1,4,11	13. (Ma & Tab)	9,11	&I
4,11	14. (∃y) (Fy & Tyb)	12	EI
1,4,11	15. (∃y) (My & Tyb)	13	EI
1,4,11	16. ((∃y) (Fy & Tyb) & (∃y) (My & Tyb))	14,15	&I
1,4,11	17. (∃x) ((∃y) (Fy & Tyx) & (∃y) (My & Tyx))	16	&I
1,3,4	18. (∃x) ((∃y) (Fy & Tyx) & (∃y) (My & Tyx))	10,11,17	EE
1,2,3	19. (∃x) ((∃y) (Fy & Tyx) & (∃y) (My & Tyx))	2,4,18 EE	

Lecture 31

For predicate logic, we have rules from propositional logic, plus two rules for each of the quantifiers. Now, we add final rules, for =.

Identity Introduction (=I): for any constant c, we get c = c from no assumptions.

Identity Elimination (=E): For any constants c and c', if A is a sentence and A' is the result of replacing some occurrences of c in A by c', then from A and c = c', we may obtain A'. The lines referred to are the one where we have A and the one where we have c = c'. The assumptions are the ones behind these lines. As for all the other rules, we need a perfect match. It is not good enough to have A and c' = c, where c appears in A.

<u>Example 1</u>: a = b ⊢ b = a

1	1. a = b	A		
	2. a = a	=I		
1	3. b = a	1,2	=E	

<u>Example 2</u>: a = b, b = c ⊨ a = c

1	1. a = b	A		
2	2. b = c	A		
1,2	3. a = c	1,2	=E (we are replacing the occurrence of b in 1 by c)	

<u>Example 3</u>: ⊢ (x) (y) ((Fx & x = y) → Fy)

1	1. (Fa & a = b)		A
1	2. Fa 1	1	&E
1	3. a = b	1	&E
1	4. Fb	2,3	=E
	5. ((Fa & a = b) → Fb)	1,4	CP
	6. (y) ((Fa & a = y)→ Fy)	5	UI
	7. (x) (y) ((Fx & x = y) → Fy)	6	UI

p. 167: 2(b) Translate and then give proof.

No murderers are sane; Jekyll is a murderer; Hyde is sane; therefore, Jekyll is not Hyde

Predicates: M-murderer; S-sane; Constants: j-Jekyll; h-Hyde

(x) (Mx → ¬Sx), Mj, Sh ⊢ ¬j = h

1	1. (x) (Mx → ¬Sx)	A
2	2. Mj	A

3	3. Sh		A	
4	4. j = h		A (use RAA)	
1	5. (Mj → ¬Sj)	1	UE	
1,2	6. ¬Sj	5,2	MPP	
1,2,4	7. ¬Sh	6,4	EI	
1,2,3,4	8. (Sh & ¬Sh)	3,7	&I	
1,2,3	9. ¬j = h		4,8	RAA

4 (a) Translate: there are at most 2 things with property F.

(∃x) (∃y) (z) (Fz → z = x ∨ z = y)

1(a) Fa -||- (x) (x = a → Fx)

First, show Fa |- (x) (x = a → Fx)

1	1. Fa A			

Show (b = a → Fb)

2	2. b = a			A (if we had a = b, we could use =E to get Fb)
3	3. ¬Fb		A (use RAA)	
2,3	4. ¬Fa	3,2	=E	
1,2,3	5. (Fa & ¬Fa)	1,4	&I	
1,2	6. ¬¬Fb		3,5	RAA
1,2	7. Fb		6	DN
1	8. (b = a → Fb)		2,7	CP
1	9. (x) (x = a → Fx)		8	UI

Now, show (x) (x = a → Fx) |- Fa

1	1. (x) (x = a → Fx)		A
1	2. (a = a → Fa)	1	UE
	3. a = a		=I
1	4. Fa	2,3	MPP

Note on (d): a = b |- (Fa ↔ Fb) (abbr. for (Fa → Fb) & (Fb → Fa)); similarly on (e).

(f) |- (∃x) x = a

1	1. ¬(∃x) x = a		A (use RAA)	
2	2. b = a			A (use RAA)
2	3. (∃x) x = a		2	EI
1,2	4. ((∃x) x = a & ¬(∃x) x = a)	3,1	&I	
1	5. ¬b = a		2,4	RAA
1	6. (x) ¬x = a		1	UI
1	7. ¬a = a		6	UE
	8. a = a			=I

1	9. (a = a & ¬a = a)	8,7	&I
1	10. ¬¬(∃x) x = a	1,9	RAA
1	11. (∃x) x = a	10	DN

Homework (not to turn in): p. 167 1(b),(c),(d); 2(a),(c); 4(b).

Solutions to problems on p. 167

1(b) |- (x) (y) ((Fx & x = y) → Fy)

1	1. (Fa & a = b)		A
1	2. Fa	1	&E
1	3. a = b	1	&E
1	4. Fb	2,3	=E
	5. ((Fa & a = b) → Fb)	1,4	CP
	6. (y) ((Fa & a = y) → Fy)	5	UI
	7. (x) (y) ((Fx & x = y) → Fy)	6	UI

(c) b = a, c = a |- b = c

1	1. b = a		A
2	2. c = a		A
	3. c = c		=I
2	4. a = c	3,2	=E
1,2	5. b = c	1,4	=E

(d) a = b |= (Fa ↔ Fb) (this is an abbreviation for ((Fa → Fb) & (Fb → Fa))

1	1. a = b		A
2	2. Fa		A (use CP)
1,2	3. Fb	2,1	=E
1	4. (Fa → Fb)	2,3	CP
5	5. Fb		A (use CP)
	6. a = a		=I
1	7. b = a	6,1	=E
1,5	8. Fa	5,7	=E
1	9. (Fb → Fa)	5,8	CP
1	10. ((Fa → Fb) & (Fb → Fa))	4,9	&I

2(a) All murderers are insane; Jekyll is a murderer; Jekyll is Hyde; therefore, Hyde is insane.
predicates: M--murderer; I--insane; constants: j--Jekyll; h--Hyde.
(x) (Mx → Ix), Mj, j = h |= Ih

1	1. (x) (Mx → Ix)		A
2	2. Mj		A
3	3. j = h		A
1	4. (Mj → Ij)	1	UE
1,2	5. Ij	4,2	MPP
1,2,3	6. Ih	5,3	=E

(b) Only Tom and Jane are dancing; Tom and Jane are doing the twist; therefore, everyone dancing is doing the twist.
constants: t--Tom; j--Jane; predicates: D--dancing; T--doing the twist.
(x) (Dx → (t = x ∨ j = x)), (Tt & Tj) |- (x) (Dx → Tx)

1	1. (x) (Dx → (t = x ∨ j = x))		A
2	2. (Tt & Tj)		A
3	3. Dc		A (prove Tc and use CP)
1	4. (Dc → (t = c ∨ j = c))	1	UE
1,3	5. (t = c ∨ j = c)	4,3	MPP
6	6. t = c		A (use ∨E--start of first case)
2	7. Tt	2	&E
2,6	8. Tc	7,6	=E
9	9. j = c		A (use ∨E--start of second case)
2	10. Tj	2	&E
2,9	11. Tc	10,9	=E
1,2,3	12. Tc	5,6,8,9,11	∨E
1,2	13. (Dc → Tc)	3,13	CP
1,2	14. (x) (Dx → Tx)	13	UI

4(b) There are exactly three things with property F. Or, there exist three distinct things such that a thing has property F just in case it is one of these three.
(∃x) (∃y) (∃z) (((¬x = y & ¬x = z) & ¬y = z) & (u) (Fu ↔ ((u = x ∨ u = y) ∨ u = z))

Solutions to problems from old exam. III.

6. (i) translates as (c); (ii) translates as (b); (iii) translates as (iv); (iv) translates as (e); (v) translates as (d).

7.

1	1. Pm	A		
1	2. (x) Px	1	UI	

Line 2 is not a proper use of UI since the constant m occurs in the assumption behind line 1 (which is line 1 itself).

8. (a) (x) Ax, (x) (Ax → Bx) |- (x) Bx

1	1. (x) Ax		A
2	2. (x) (Ax → Bx)		A
1	3. Ac	1	UE
2	4. (Ac → Bc) 2	UE	
1,2	5. Bc	4,3	MPP
1,2	6. (x) Bx	5	UI

(b) (x) Ax |- (y) (Ay ∨ By)

1	1. (x) Ax		A
1	2. Ac	1	UE
1	3. (Ac ∨ Bc)	2	∨I
1	4. (y) (Ay ∨ By)	3	UI

(c) (x) (y) Lxy |- Lmn

1	1. (x) (y) Lxy	A		
1	2. (y) Lmy	1	UE	
1	3. Lmn	2	UE	

(d) ⊢ (x) ((Ax & Bx) → Ax)

1	1. (Ac & Bc)			A (prove Ac & use CP)
1	2. Ac		1	&E
	3. ((Ac & Bc) → Ac)		1,2	CP
	4. (x) ((Ax & Bx) → Ax)		3	UI

Math. 112 student talks, Room 282, Galvin Life Sciences

<u>April 16</u>
Larice Woods (Fuzzy Logic), April Olsen (Intuitionism), Brigid Reagan (Intuitionism)

<u>April 18</u> (Brigid Reagan, chair)
Matthew Mahoney (Absurdities), Neil O'Connor (Infinite Divisibility),
Megan Paulsen (Paradoxes)

<u>April 21</u> (April Olsen, chair)
John Daily, Kyle Smith, Douglas Seckenger (Paradoxes)

<u>April 23</u> (Larice Woods, chair)
Emily Reimer (Paradoxes and the development of set theory), Jeremy Lyons (Cantor's set theory), Chiara Fortino (Sizes of sets--from finite to infinite)

<u>April 25</u> (Neil O'Connor*, chair)
Jonathan Kaminski (Critique of text), Kathryn Catenacci (Completeness and soundness),
Kenneth Macek (Gö del's Incompleteness Theorems)

<u>April 28</u> (Matt Mahoney*, chair)
Eric Francis, A. J. Boyd, Kevin Robinson (Derived rules of proof).

It is important that you attend <u>all</u> of the talks. Try to be responsive listener. Think about the material, ask questions, and let the speaker know if the presentation is good, or the topic seems interesting.

You may be asked to chair a session of talks.

The duties of the chair are to introduce each speaker and give the title of the talk, to offer assistance with the projector, etc., to keep time, and at the end of the talk (time permitting), to call for questions or comments.

Please sign up for one of the rehearsal times listed below. Give your name and the title of your talk. The rehearsals will be in room 214, MCC.

April 14, morning session

8:00-8:15

8:20-8:35

8:40-8:55

9:00-9:15

April 14, afternoon session

3:15-3:30

3:35-3:50

3:55-4:10

4:15-4:30

4:35-4:50

April 15, morning session

8:00-8:15

8:20-8:35

8:40-8:55

9:00-9:15

9:20-9:35

April 15, afternoon session

3:15-3:30

3:35-3:50

3:55-4:10

4:15-4:30

Lecture 32

Hand out homework, solutions to problems on =.

Hand out schedule of talks, rehearsals.

Recall Exam III, Friday in class. Thursday evening help session 7-9 in Room 214 MCC.

Exam. is mainly proofs, some translation. Some proofs involve =. Not very long or tricky.

More examples involving = ?

Example 1: |- (x) (y) ((Fx & x = y) → Fy)

1	1. (Fa & a = b)		A
1	2. Fa	1	&E
1	3. a = b	1	&E
1	4. Fb	2,3	=E
	5. ((Fa & a = b) → Fb)	1,4	CP
	6. (y) ((Fa & a = y)→ Fy)	5	UI
	7. (x) (y) ((Fx & x = y) → Fy)	6	UI

Example 2: 1(a) Fa -||- (x) (x = a → Fx)

First, show Fa |- (x) (x = a → Fx)

1 1. Fa A
Show (b = a → Fb)

2	2. b = a		A (if we had a = b, we could use =E to get Fb)
3	3. ¬Fb	A (use RAA)	
2,3	4. ¬Fa 3,2	=E	
1,2,3	5. (Fa & ¬Fa) 1,4	&I	
1,2	6. ¬¬Fb	3,5	RAA
1,2	7. Fb	6	DN
1	8. (b = a → Fb)	2,7	CP
1	9. (x) (x = a → Fx)	8	UI

Now, show (x) (x = a → Fx) |- Fa

1	1. (x) (x = a → Fx)		A
1	2. (a = a → Fa)	1	UE
	3. a = a		=I
1	4. Fa	2,3	MPP

Example 3: 1 (f) |- (∃x) x = a

1	1. ¬(∃x) x = a		A (use RAA)
2	2. b = a		A (use RAA)
2	3. (∃x) x = a	2	EI
1,2	4. ((∃x) x = a & ¬(∃x) x = a)	3,1	&I
1	5. ¬b = a	2,4	RAA
1	6. (x) ¬x = a	1	UI
1	7. ¬a = a	6	UE
	8. a = a		=I
1	9. (a = a & ¬a = a)	8,7	&I
1	10. ¬¬(∃x) x = a	1,9	RAA
1	11. (∃x) x = a	10	DN

Class 33, Exam. III

Solutions to Exam. III

1.

1	1.	$(x)(Fx \to Gx)$		A
2	2.	Fc		A
1	3.	$(Fc \to Gc)$	1	UE
1,2	4.	Gc	3,2	MPP

2.

1	1.	$(x)(y)Hxy$		A
1	2.	$(y)Hay$	1	UE
1	3.	Hab	2	UE
1	4.	$(x)Hxb$	3	UI
1	5.	$(y)(x)Hxy$	4	UI

3.

1	1.	$(\exists x)Fx$		A
2	2.	$(x)\neg Gx$		A
3	3.	Fa		A (use EE)
2	4.	$\neg Ga$	2	UE
2,3	5.	$(Fa \,\&\, \neg Ga)$	3,4	&I
2,3	6.	$(\exists x)(Fx \,\&\, \neg Gx)$	5	EI
1,2	7.	$(\exists x)(Fx \,\&\, \neg Gx)$	1,3,6	EE

4.

1	1.	$(Fa \vee Fb)$		A
2	2.	Fa		A (use ∨E)
2	3.	$(\exists x)Fx$	2	EI
4	4.	Fb		A (use ∨E)
4	5.	$(\exists x)Fx$	4	EI
1	6.	$(\exists x)Fx$	1,2,3,4,5	∨E

5.

	1.	$a = a$		=I
	2.	$(x)\, x = x$	1	UI

6.

1	1.	$a = b$		A
2	2.	Lab		A (use CP)
1,2	3.	Lbb	2,1	=E
1	4.	$(Lab \to Lbb)$	2,3	CP

7. (i) matches (d)
   (ii) matches (b)
   (iii) matches (a)
   (iv) matches (c)

8. Line 3 is not justified. The constant in the line referred to (2) appears in the assumption behind that line (2), so it is not arbitrary.

Class 34

Scale: A: 85↑
B: 70↑
C: 60↑
D: 50↑

Hand out exams and solutions.
Probably the exam. should have been a bit longer so that missing one would not cost so much. Questions ?

Talks--rehearsals in Room 214 MCC. You may rehearse first and then prepare slides, but you should test your slides.

Needn't have abstracts when you rehearse.

We have all of the primitive rules of proof for predicate logic, including rules for equality.

There are Soundness and Completeness theorems for this set of rules. As for propositional logic, understanding these results requires notion of truth.

The history of the formal notion of truth is interesting.

Gö del proved the Completeness theorem in his Ph.D. thesis, so he had to have a definition. But, he didn't discuss it. "In a crossed-out passage of an unsent reply to a graduate student's later inquiry, Gö del asserted that at the time of his completeness and incompleteness papers, 'a concept of objective mathematical truth...was viewed with [the] greatest suspicion and [was] widely rejected as meaningless'.

Tarski was the one who finally published the formal definition of truth. But before that, when Tarski had proved some important results related to the notion, and Carnap asked him to talk about it at a philosophical meeting, Carnap says, "Tarski was very skeptical. He thought that most philosophers, even those working in modern logic, would be not only indifferent, but hostile to the explication of the concept of truth." And, Carnap continues, "in fact, Tarski's skeptical predictions had been right.....there was vehement opposition...."

First, we say what is a <u>structure</u> for a particular language.

I. Suppose our language has two 1-placed predicate symbols F and G, no other predicate symbols. A structure for this language consists of a non-empty set, the <u>universe</u>, with subsets to interpret F and G. Formally, the structure has the form (U,F',G'), where U is the universe, and F', G' are the interpretations of F and G.

For example, take (U,F',G'), where U is the set of all students at ND, F' is the set of freshmen, and G' is the set of students currently enrolled in Math. 112.

Or, we may let the universe be the set of all natural numbers, and interpret F as the set of even numbers and G as the set of odd numbers.

II. Suppose the language has one 2-placed predicate symbol L and one constant m. A structure for this language consists of a non-empty set, the universe, with a 2-placed relation to interpret L and an element to interpret the constant. Formally, the structure has the form (U,L',m').

For example, take (N,<,0), where N, the universe, is the set of natural numbers 0, 1, 2, ..., <, the interpretation of L, is the usual ordering, and 0 is the interpretation of m.

Or, let U be the set of all people, interpret L as "likes", and interpret m as Mother Theresa.

Now, satisfaction and truth.

We say what it means for a formula $A(x_1,...,x_n)$, with free variables among $x_1,...,x_n$, to be satisfied in an appropriate structure when elements $a_1,...,a_n$ are sutstituted for $x_1,...,x_n$.

If A is a sentence, and it is satisfied in the structure, we say that it is true.

The definition of satisfaction is so natural, it is embarrassing to write it all out.

Consider the language with the 2-placed predicate symbol L. Take the structure (U,L'), where U is set of natural numbers 0, 1, 2, ..., L'ab holds just in case a < b.

Formula (∃y) Lyx says that there is somethng less than x. This is satisfied by 1, 2, 3, etc., not by 0.

Sentence (x) (∃y) Lxy is true--for any number, there is one which is larger.

Formula (Lxy → (∃z) (Lxz & Lzy)) says that if x is less than y, then there is something in between. This is satisfied if we substitute 1 for x and 3 for y, or 3 for x and 2 for y, not if we substitute 1 for x and 2 for y. The sentence (x) (y) (Lxy → (∃z) (Lxz & Lzy)), saying that for any two numbers, if x is less than y, there is something in between, is not true.

The sentence (x) (y) ((Lxy ∨ Lyx) ∨ x = y), saying that for any numbers x and y, either one is less than the other, or the two are equal, is true.

Take the structure (U,L'), where U is the set of all people, and L'ab holds just in case a likes b. Then (y) Lxy is satisfied by those people who like everyone.

Fix a predicate language. Consider sentences and structures for this language. We write $A_1,...,A_n \models B$ if for any structure where $A_1,...,A_n$ are all true, B is also true. Now, the theorems look the same as for propositional logic.

<u>Soundness</u>: If $A_1,...,A_n$ |- B, then $A_1,...,A_n$ |= B.

<u>Completeness</u>: If $A_1,...,A_n$ |= B, then $A_1,...,A_n$ |- B.

For propositional logic, we could look at all truth assignments for a particular formula or finite set of formulas, forming a truth table. We had a method for deciding whether |= B, or $A_1,...,A_n$ |= B. For predicate logic, there is nothing like this. In propositional logic, it principal, we can determine whether there is a proof, and if there is one, we will eventually be able to find it. In predicate logic, we can try to find a proof, but we cannot tell whether there is one. In principal, for an argument in propositional logic, a machine can determine whether it is sound. For an argument in predicate logic, this is not so. Unless we see the proof, we do not know whether the argument is sound.

Wednesday, start talks based on papers.

There will be no homework to turn in. Final will cover material from in-class exams., plus one question on talks: Which topic, other than your own, strikes you as most interesting. I will try to prepare some sample problems for you to use in reviewing for the final.

Final exam. Monday, May 5, 8:00-10:00, Room 120 De Bartolo

Help session Sunday, 7:00-9:00, Room 214, Math. Building
I will miss office hour today, should be in tomorrow 8-9:30, also later in day (unpredictable), Friday until 4:00.

Final is cumulative. Questions ?

Course has covered the following.

Propositional logic: wff's (definition), translation between English and formal statements. Rules of proof. Truth, truth tables, classifying statements as tautologous, contingent, inconsistent, deciding whether arguments are sound. Statements of Soundness and Completeness.

Predicate logic: wff's (definition), translation between English and formal statements. Rules of proof. I indicated briefly that for predicate logic as well as for propositional logic, there is natural definition of truth, satisfaction. With this definition, Soundness and Completeness results hold.

Material from talks.

There are many other topics in logic, some in mathematics, some in philosophy, some in computer science. Can we talk sensibly about the size of a set which is not finite ? Are there limits to what we can analyze formally ? What is computable ? Some of these were discussed in talks. Some talks pointed out connections between logic and other areas. In particular, there were connections between paradoxes in logic, and paradoxes in language. Mahoney connected our RAA with absurdity in literature and everyday life.

Our rules for forming formulas (our grammar), our truth definitions, and our rules of proof were "classical". There are other grammars, other notions of truth, other kinds of logic.

I hope that feel you feel you have gotten something out of the course. Possibly, you will find it useful in other contexts to translate statements or analyze arguments formally. Or, without being completely formal, you will use idea of breaking complex statements into simpler parts. The process you went through in writing your paper and giving a talk is, I hope, a useful exercise. I believe in the breadth requirements. You cannot possibly predict at this stage exactly what you need to know for your future job. The best preparation, I feel, is to develop strategies for learning a variety of kinds of material. Some of you indicated that you didn't much like calculus. Nonetheless, you were willing to work on the technical material in this course, and really master the rules of proof. The papers and talks were very good, very creative. I wonder what would happen if instead of drill problems in calculus, you were given a paper to write on the idea of velocity at an instant.

I have really enjoyed having you as students.

Ratings.