Exam I
September 25, 2001

11. The formula from Newton's method is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ which in our case amounts to

$$x_1 = x_0 - \frac{x_0^3 + x_0^2 + 1}{3x_0^2 + 2x_0} .$$

Compute $f(-2) = (-2)^3+(-2)^2+1 = -8+4+1 = -3 < 0$ and $f(-1) = (-1)^3+)-1)^2+1 = -1+1+1 = 1 > 0$ so by the Intermediate Value Theorem, there is at least one zero in this interval. (**Parenthetical Remark**: It is not hard to check that $3x^2 + 2x > 0$ on the interval $[-2, -1]$ so f is increasing on this interval and so there is precisely one zero there.)

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12. If the two sides of the rectangle are x and y, then the area is $A = xy$ and the cost of fence is $C = 10x + 20(x+2y) = 30x + 40y$ if we let x denote the length of the side with the fence. Since $A = 80$, $y = \frac{80}{x}$ and $C = 30x + 40 \cdot \frac{80}{x}$. We need to minimize C so compute $\frac{dC}{dx} = 30 - 40 \cdot \frac{80}{x^2}$. The critical points are $x = 0$ and $30 - 40 \cdot \frac{80}{x^2} = 0$, or $40 \cdot \frac{80}{x^2} = 30$, or $x^2 = \frac{40 \cdot 80}{30} = \frac{320}{3}$. Hence $x = \pm\sqrt{\frac{320}{3}}$. In our problem, $x > 0$ so $x = \sqrt{\frac{320}{3}}$ is a critical point. By the first derivative test, $x = \sqrt{\frac{320}{3}}$ is a local maximum and is the only critical point on the interval $(0, \infty)$ and so there it is a global minimum.

Hence you should make the side with the fence $\sqrt{\frac{320}{3}}$ feet long and the other side of the rectangle should be $\frac{80}{\sqrt{\frac{320}{3}}}$ feet long. The total cost in dollars is $30 \cdot \sqrt{\frac{320}{3}} + 40 \cdot \frac{80}{\frac{80}{\sqrt{\frac{320}{3}}}} = 30 \cdot \sqrt{\frac{320}{3}} + 40 \cdot \sqrt{\frac{320}{3}} = 70 \cdot \sqrt{\frac{320}{3}}$.

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13. You are being asked to minimize the distance from the point (x, x^2) on the parabola to the point$(0, 1)$. A formula for this distance is $d = \sqrt{(x-0)^2 + (x^2-1)^2}$. You can simplify your work by minimizing d^2 but we will stick with d. Well, $d = \sqrt{x^2 + x^4 - 2x^2 + 1} = \sqrt{x^4 - x^2 + 1}$. The only critical points in this case occur where the derivative, $\frac{4x^3 - 2x}{2\sqrt{x^2+x^4-2x^2+1}}$, vanishes. But this happens if and only if $4x^3 - 2x = 0$ or when $x = 0$ and $x = \pm\sqrt{\frac{1}{2}}$. The sign of the derivative alternates as we pass through each critical point, so $\pm\sqrt{\frac{1}{2}}$ are local minima and 0 is a local maximum. Since $\lim_{x \to \pm\infty} d = \infty$, it follows that $x = \pm\sqrt{\frac{1}{2}}$ are global minima. The corresponding points on the parabola are $\left(\pm\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$.

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14. Since $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ in general we proceed as follows. In our case, $a=0$, $b=1$ and $f(x)=3x^2+2$. Moreover, x_i^* is to be the right–hand end point of the ith interval. Then $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ and the right–hand end point of the ith interval is given by $x_i^* = a + i\Delta x = \frac{i}{n}$.

Hence we get $\displaystyle\int_0^1 (3x^2+2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(\frac{i}{n}\right)^2+2\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3i^2}{n^3}+\frac{2}{n}\right) =$ $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3i^2+2n^2}{n^3}\right) = \lim_{n\to\infty}\frac{3\sum_{i=1}^{n} i^2 + 2n^2\sum_{i=1}^{n} 1}{n^3} = \lim_{n\to\infty}\frac{3\frac{n(n+1)(2n+1)}{6}+2n^3}{n^3} =$ $\displaystyle\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{2n^3} + \lim_{n\to\infty} 2 = 1+2 = 3.$