Math. 125 Quiz #1 September 4, 2001

1. If

$$(f \circ g)(x) = f\big(g(x)\big) = \sin(x^2 + 1)$$

what what is a possibility for f and g?

2. If $h(t) = \dfrac{t}{t+1}$ and $m(s) = 2s + 3$ what is $(h \circ m)(y) = h\big(m(y)\big)$?

***************** Answer(s) *****************

1. One choice is $f(x) = \sin(x)$ and $g(x) = x^2 + 1$. Another option is $f(x) = \sin(x+1)$ and $g(x) = x^2$.

2. $(h \circ m)(y) = \dfrac{2y+3}{2y+4}$.

Math. 125 Quiz #2 September 11, 2001

1. What is $\displaystyle\lim_{x \to -1} \frac{2+x}{1-x}$?

2. What is $\displaystyle\lim_{x \to 0^+} \frac{1}{x}$?

3. Suppose $\displaystyle\lim_{x \to 1} f(x) = 4$ and $\displaystyle\lim_{x \to 1} g(x) = -2$. If the following limit exists, what is it?

$$\lim_{x \to 1} \frac{2 \cdot f(x) + 3 \cdot g(x)}{\big(8 + f(x) \cdot g(x)\big)^2}$$

4. What is $\displaystyle\lim_{x \to 3} x^2 - 2x - 3$?

***************** Answer(s) *****************

1. $\dfrac{2 + (-1)}{1 - (-1)} = \dfrac{1}{2}$ (just plug in).

2. $+\infty$. This is a basic limit you should know.

3. Using various theorems we get $\dfrac{2 \cdot 4 + 3 \cdot (-2)}{\big(8 + 4 \cdot (-2)\big)^2} = \dfrac{2}{0}$ so our theorems are not immediately applicable. However, the top of this fraction is close to 2 where as the bottom is close to 0. Because of the square in the denominator, not only is the denominator close to 0, but it is also positive so the answer is $+\infty$.

4. Just plug in again $3^3 - 2 \cdot 3 - 3 = 0$.

Math. 125 Quiz #3 September 18, 2001

1. Show $f'(x) = 3x^2 + 3$ using the definition of the derivative where

$$f(x) = x^3 + 3x + 2 \ .$$

2. Find all the vertical tangents of f.

***************** Answer(s) *****************

1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\big((x+h)^3+3(x+h)+2\big)-\big(x^3+3x+2\big)}{h} \\ &= \lim_{h\to 0} \frac{\big(x^3+3x^2h+3xh^2+h^3+3x+3h+2\big)-\big(x^3+3x+2\big)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3+3h}{h} = 3x^2+3xh+h^2+3 = x^2+3\ . \end{aligned}$$

2. The only place where a vertical tangent can occur is when $\lim_{x\to a^\pm} f'(x) = \pm\infty$. But $f'(x)$ is a polynomial and hence continuous everywhere so there are no vertical tangents.

Math. 125 Quiz #4 October 2, 2001

1. Find the derivative of

$$f(x) = \sin\big(\cos(x^2)\big)$$

2. Find all values of x for which the tangent line is horizontal.
Remark: $\frac{\pi}{2} > 1$.

***************** Answer(s) *****************

1. By the chain rule $f'(x) = \cos\big(\cos(x^2)\big) \cdot \big(-\sin(x^2)\big) \cdot (2x)$.

2. We need to solve $f'(x) = 0$, so first "simplify" our expression for $f'(x)$: $f'(x) = (-2x) \cdot \big(\sin(x^2)\big) \cdot \cos\big(\cos(x^2)\big)$. The only way a product can vanish is for one of the factors to vanish so we need to solve $(-2x) = 0$, $\sin(x^2) = 0$ and $\cos\big(\cos(x^2)\big) = 0$. The frist equation has the unique solution $x = 0$ and the hint plus the facts that $-1 \le \cos(x) \le 1$ and $\cos(x) = 0$ only when $x = \frac{\pi}{2} + k\pi$ for integers k, shows that the third equation has no solutions. This leaves $\sin(x^2) = 0$. Well, $\sin(x) = 0$ if and only if $x = k\pi$, k any integer, so we need to solve $x^2 = k\pi$, and therefore $x = \pm\sqrt{k\pi}$ for k a non–negative integer. The solution $x = 0$ is included in the set $x = \pm\sqrt{k\pi}$ for k a non–negative integer so these are all the solutions and hence all the horizontal tangents.

Math. 125 Quiz #5 October 9, 2001

1. Car A and car B are approaching the intersection "C" of two streets intersecting at a right angle. Car A is going South at 45 mph, car B is heading West at 30 mph. We denote the angle $\angle(C, B, A)$ by θ, the distance from C to B by x, and the distance from C to A by y. Then, $\tan\theta = \frac{y}{x}$. At what rate is the angle θ changing when car A and car B are both 1 mile from the intersection?

q5.eps

***************** Answer(s) *****************

We are asked to compute $\frac{d\theta}{dt}$ when $x = 1$ and $y = 1$. We are given a relation $\tan\theta = \frac{y}{x}$ which holds all the time, so differentiating

$$(*) \qquad (\sec^2\theta)\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

We are further given $\frac{dy}{dt} = -45$ and $\frac{dx}{dt} = -30$ and we can plug onto $(*)$ provided we can compute $\sec^2\theta$ when $x = y = 1$. When $x = y = 1$, $\tan\theta = 1$ so $\sec^2\theta = 1 + \tan^2\theta = 2$ and we get $2\frac{d\theta}{dt} = (-45) - (-30) = -15$ so $\frac{d\theta}{dt} = -\frac{15}{2}$.

Math. 125 Quiz #6 October 16, 2001

Find all the critical points for the function below. Which ones are local maxima? Why? Which ones are local minima? Why?
The existence theorem for global max/min does not apply to this function restricted to the interval $(-\infty, -2)$ since the interval must be a closed interval for the theorem to apply. Nevertheless, this function does have a minimum value on this interval. Where is it and why is it where you say? Why does this function have no maximum value on this interval?

$$\frac{(x-5)^8}{(x+2)^4}$$

***************** Answer(s) *****************

First we need to compute the derivative of $r(x) = \frac{(x-5)^8}{(x+2)^4}$:

$$\begin{aligned} r'(x) =& \frac{8(x-5)^7(x+2)^4 - (x-5)^8 4(x+2)^3}{(x+2)^8} = (x-5)^7(x+2)^3\frac{8(x+2) - (x-5)4}{(x+2)^8} \\ =& \frac{(x-5)^7\big(8x+16-4x+20\big)}{(x+2)^5} = \frac{(x-5)^7(4x+36)}{(x+2)^5} = \frac{4(x-5)^7(x+9)}{(x+2)^5} \ . \end{aligned}$$

To find the critical points, note that $r'(x)$ is continuous everywhere it exists so we only need to solve $r'(x) = 0$ and find all the points where r' does not exist. Well, $r'(x) = 0$ only when $(x-5)^7(x+9) = 0$ so $x = 5$ and $x = -9$: $r'(x)$ does not exist precisely when $(x+2)^5 = 0$ so $x = -2$.

−	+	−	+
	-9	-2	5

Local minima occur at $x = -9$ and $x = 5$ because the derivative changes from negative to positive at these points. The value $x = -2$ is not a local extremum since the function is not defined there. To understand $r(x)$ on the interval $(-\infty, -2)$ it helps to compute $\lim_{x\to\infty} r(x) = +\infty$ and $\lim_{x\to -2^-} r(x) = +\infty$. Hence $r(x)$ starts out very big far to the left and decreases until $x = -9$. Thereafter, $r(x)$ steadily increases, getting larger and larger. Hence, if $x < -9$, $f(x) > f(-9)$ and if $x > 9$, $f(x) > f(9)$ which is the definition of an absolute minimum. There can be no maximum on the interval $(-\infty, -2)$ since $r(x)$ gets larger and larger as x gets closer and closer to -2.

Math. 125 Quiz #7 November 6, 2001

Find the rectangle of largest area that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

.

q7.eps

****************** Answer(s) ******************

This is a max.-min. problem: specifically, we are being asked to maximize the area of a rectangle. The sides of the rectangle are $2x$ and $2y$, so the area is $A = 4xy$. This is typical of many max.-min. problems in that A is a function of two variables, so we must eliminate one of them. The required equation comes from the fact that the point (x, y) lies on the ellipse, so $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Solve for $y = b\sqrt{1 - \frac{x^2}{a^2}}$. (Since our point is in the first quadrant, we want the solution for which $y \geq 0$.) Hence $A = 4bx\sqrt{1 - \frac{x^2}{a^2}}$ is now a function of one variable. Additionally, we have set things up so that $0 \leq x \leq a$.

The next step is to locate the critical points of $A(x)$ so calculate

$$A'(x) = 4b\left(\sqrt{1 - \frac{x^2}{a^2}} + x\frac{\frac{-2x}{a^2}}{2\sqrt{1 - \frac{x^2}{a^2}}}\right) = 4b\left(\frac{1 - \frac{x^2}{a^2} - \frac{x^2}{a^2}}{\sqrt{1 - \frac{x^2}{a^2}}}\right) = 4b\left(\frac{1 - 2\frac{x^2}{a^2}}{\sqrt{1 - \frac{x^2}{a^2}}}\right)$$

The critical points occur where A' is not defined ($x = a$) or where $A' = 0$ (or $x = \frac{a}{\sqrt{2}}$). To locate the maximum, note $A(0) = A(a) = 0$ and $A(\frac{a}{\sqrt{2}}) = 2ab$ so the largest area rectangle occurs when $(x, y) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

Math. 125 Quiz #8 November 13, 2001

We wish to use Newton's method to find a solution to the equation

$$\sec x = \tan x\ .$$

Of course this is equivalent to solving the equation $\sec x - \tan x = 0$. If we make a first approximation of $x = 0$, what does Newton's method give for a second approximation?

****************** Answer(s) ******************

Since $f(x) = \sec x - \tan x$, $f'(x) = \sec x \tan x - \sec^2 x$, so we get

$$x_1 = 0 - \frac{f(0)}{f'(0)} \ .$$

Since $f(0) = \sec(0) - \tan(0) = 1$, $f'(0) = \sec(0)\tan(0) - \sec^2(0) = 1 \cdot 0 - 1^2 = -1$ so $x_1 = 1$.

Math. 125 Quiz #9 November 20, 2001

1. Consider the region in the plane below the graph $y = \sin x$; above the x–axis and between the vertical lines $x = 0$ and $x = 2$.

a. Divide the interval into 6 equal parts and write down the Riemann sum using the left–hand end points of each interval. Do not evaluate the sum or even try to simplify it.

b. Write a definite integral which evaluates the area precisely. Do not evaluate the integral.

2. What number is represented by the sum

$$\sum_{q=-1}^{2} q(q^2 - 1) \ ?$$

****************** Answer(s) ******************

1a. The width of each interval is $\dfrac{2-0}{6} = \frac{1}{3}$; the left hand end points are 0, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3} = 1$, $\frac{4}{3}$ and $\frac{5}{3}$. Hence the asked for Riemann sum is
$\sin(0) \cdot \frac{1}{3} + \sin(\frac{1}{3}) \cdot \frac{1}{3} + \sin(\frac{2}{3}) \cdot \frac{1}{3} + \sin(1) \cdot \frac{1}{3} + \sin(\frac{4}{3}) \cdot \frac{1}{3} + \sin(\frac{5}{3}) \cdot \frac{1}{3}$.

1b. $\displaystyle\int_0^2 \sin x \, dx$.

2. The sum is $(-1)\big((-1)^2 - 1\big) + (0)\big((0)^2 - 1\big) + (1)\big((1)^2 - 1\big) + (2)\big((2)^2 - 1\big)$. All the terms but the last vanish so the number is $2(4-1) = 6$.

Math. 125 Quiz #10 November 27, 2001

Find the derivative of the function

$$G(x) = \int_0^{\int_1^x \frac{1}{t} dt} \csc\theta \ d\theta$$

with respect to x.

****************** Answer(s) ******************

By the Chain Rule and the Fundamental Theorem of Calculus,

$$G'(x) = \csc\left(\int_1^x \frac{1}{t}dt\right) \cdot \frac{d\int_1^x \frac{1}{t}dt}{dx} \ .$$

We can do nothing with the integral inside the csc, but we can apply the Fundamental Theorem again to compute $\frac{d\int_1^x \frac{1}{t}dt}{dx} = \frac{1}{x}$, so

$$G'(x) = \csc\left(\int_1^x \frac{1}{t}dt\right) \cdot \frac{1}{x} \ .$$

Math. 125 Quiz #11 November 11, 2001

1. Write a definite integral that evaluates to the area of the shaded region below. Do not evaluate the integral.

2. Rotate the shaded region about the line $y = 2$ and write a definite integral that evaluates to the volume of the resulting solid of revolution. Do not evaluate the integral.

3. Rotate the shaded region about the line $x = 3$ and write a definite integral that evaluates to the volume of the resulting solid of revolution. Do not evaluate the integral.

graph11.eps

****************** Answer(s) ******************

1. It is easy to describe the two curves as graphs of y as a function of x. Hence

   Area $= \displaystyle\int_{-1}^{1} (x^2 - x^4)\, dx$.

   Having done the area, the remaining two problems are practically done.

2. Since we are rotating around $y = 2$ which is perpendicular to the x–axis, we get washers so

   $$\text{Volume}_{y=2} = \pi \int_{-1}^{1} \left((2 - x^4)^2 - (2 - x^2)^2\right)\, dx.$$

3. Since we are rotating around the line $x = 3$ which is parallel to the x–axis, we get shells so

   $$\text{Volume}_{x=3} = 2\pi \int_{-1}^{1} \left(3 - x\right)\left(x^2 - x^4\right)\, dx.$$
