Exam I September 25, 2001

11. By definition
$$f'(1) = \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \cdot \frac{\sqrt{1+h} + \sqrt{1}}{\sqrt{1+h} + \sqrt{1}}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h^2} - \sqrt{1}^2}{h} \cdot \frac{1}{\sqrt{1+h} + \sqrt{1}} = \lim_{h \to 0} \frac{1+h-1}{h} \cdot \frac{1}{\sqrt{1+h} + \sqrt{1}} \lim_{h \to 0} \frac{h}{h} \cdot \frac{1}{\sqrt{1+h} + \sqrt{1}}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + \sqrt{1}} = \frac{1}{\sqrt{1} + \sqrt{1}} = \frac{1}{2}, \text{ where the next to the last equality follows since } \sqrt{x+1} \text{ is continuous at } 0.$$

By definition
$$f'(1) = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} \cdot \frac{\sqrt{x} + \sqrt{1}}{\sqrt{x} + \sqrt{1}}$$

$$= \lim_{x \to 1} \frac{\sqrt{x^2} - \sqrt{1}^2}{x - 1} \cdot \frac{1}{\sqrt{x} + \sqrt{1}} = \lim_{x \to 1} \frac{x - 1}{x - 1} \cdot \frac{1}{\sqrt{x} + \sqrt{1}} \lim_{x \to 1} \frac{x - 1}{x - 1} \cdot \frac{1}{\sqrt{x} + \sqrt{1}}$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{x} + \sqrt{1}} = \frac{1}{\sqrt{1} + \sqrt{1}} = \frac{1}{2}, \text{ where the next to the last equality follows since } \sqrt{x} \text{ is continuous at } 0.$$

12.

- a. Since $s = 90t^{1/2} 25t^{3/2} + 3t^{5/2}, v = 90 \cdot \frac{1}{2}t^{-1/2} 25 \cdot \frac{3}{2}t^{1/2} + 3 \cdot \frac{5}{2}t^{3/2}$ or $v = \frac{90}{2}t^{-1/2} - \frac{75}{2}t^{1/2} + \frac{15}{2}t^{3/2} = \frac{15}{2}t^{-1/2}\left(6 - 5t + t^2\right). \text{ Hence } v(1) = \frac{15}{2} \cdot (6 - 5 + 1) = 15.$ b. We must solve v(t) = 0 or $0 = 6 - 5t + t^2 = (t - 3)(t - 2)$ so t = 2 and t = 3.
- c. Since there are no zeros of the velocity function between 0 and 1 the distance travelled is equal to the displacement which is s(1) - s(0) = (90 - 25 + 3) - (0) = 68.
- 13. We know f(x) is continuous at any point $x \neq 1$ since there the function is given by a polynomial and polynomials are continuous. By the same reasoning, it is easy to calculate one sided limits at x=1: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^3 + cx = 1 + c$ since $x^3 + cx$ is continuous; $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$ since x is continuous. For f to be continuous at x=1 we must have $\lim_{x\to 1} f(x) = f(1) = 1$ so 1 + c = 1 or c = 0.
- **14.** First we write down the equation for the tangent line at a: y f(a) = f'(a)(x a)and, since $f(x) = x^2 + 1$, we have f'(x) = 2x, and or line is $y - (a^2 + 1) = (2a)(x - a)$. This line passes through (0,-3) whenever $(-3)-(a^2+1)=(2a)(0-a)$. This yields $-a^2 - 4 = -2a^2$ or $a^2 = 4$ and $a = \pm 2$.

A second approach proceeds as follows. At the point $(x, x^2 + 1)$ on the curve, the slope of the tangent line is 2x and the slope of the line from this point to (0,-3) is $\frac{x^2+1-(-3)}{x-0}=\frac{x^2+4}{x}$. Equating these two slopes yields $2x=\frac{x^2+4}{x}$ which again yields $x=\pm 2$.