9. The point $(3, 2)$ lies on the curve given by the equation $y^3 + y + x^3 + x = 40$ and we wish to find $\frac{dy}{dx}$ at this point. Differentiate both sides of our equation with respect to x:

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} + 3x^2 + 1 = 0$$

We can now solve for $\frac{dy}{dx}$ and then plug in, or first plug in and then solve.

$$(*) \qquad \frac{dy}{dx}(3y^2 + 1) = -(3x^2 + 1)$$

$$\frac{dy}{dx} = -\frac{3x^2 + 1}{3y^2 + 1}$$

and plugging in yields

$$\frac{dy}{dx} = -\frac{3 \cdot 3^2 + 1}{3 \cdot 2^2 + 1} = -\frac{28}{13}$$

OR

$$3 \cdot 2^2 \frac{dy}{dx} + \frac{dy}{dx} + 3 \cdot 3^2 + 1 = 0$$

$$13 \frac{dy}{dx} + 28 = 0$$

$$\frac{dy}{dx} = -\frac{28}{13}$$

You were not asked, but notice this calculation shows that the function is decreasing at $(3, 2)$. To check the concavity at this point, it is easiest to calculate from $(*)$ by differentiating again with respect to x.

$$\frac{d^2y}{dx^2}(3y^2 + 1) + \frac{dy}{dx}(6y) = -6x$$

and plugging in $x = 3$, $y = 2$ and $\frac{dy}{dx} = -\frac{28}{13}$ yields

$$\frac{d^2y}{dx^2}(3 \cdot 2^2 + 1) - \frac{28}{13}(6 \cdot 2) = -6 \cdot 3$$

$$\frac{d^2y}{dx^2}(13) - \frac{28}{13}(12) = -18$$

$$\frac{d^2y}{dx^2}(13) = \frac{28}{13}(12) - 18$$

$$\frac{d^2y}{dx^2}(13) = \frac{28}{13}(12) - 18 = \frac{336 - 234}{13} = \frac{102}{13}$$

so

$$\frac{d^2y}{dx^2} = \frac{102}{13^2} = \frac{102}{169}$$

and the curve is concave up at this point.

10. This is a hard related rates problem in which we are asked the rate $\frac{d\gamma}{dt}$ at 2 o'clock. The first step is to observe that the lengths of the hands of the clock (a and b in our formulae) are constant: $a = 6$ $b = 4$. The other rate is $\frac{d\gamma}{dt}$ which we can compute as follows.

The hour hand revolves at a rate of $\frac{2\pi}{12} radians/hr = \frac{2\pi}{12} \cdot \frac{1}{60} radians/min.$

The minute hand revolves at a rate of $\frac{2\pi}{60} radians/min.$

Hence

$\frac{d\gamma}{dt} = \frac{2\pi}{12} \cdot \frac{1}{60} - \frac{2\pi}{60} radians/min = \frac{2\pi}{60} \cdot \frac{1}{12} - \frac{2\pi}{60} = \frac{2\pi}{60}\left(\frac{1}{12} - 1\right) = -\frac{2\pi}{60} \cdot \frac{11}{12} radians/min.$

At 2 o'clock, the hour hand is at 2, and the minute hand is straight up so $\gamma = \frac{\pi}{3} - 0 = \frac{\pi}{3} radians$ and therefore $c = \sqrt{6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cos\left(\frac{\pi}{3}\right)} = \sqrt{36 + 16 - 48 \cdot \frac{1}{2}} = \sqrt{52 - 24} = \sqrt{28}.$

The Law of Cosines holds for any triangle so we may differentiate it with respect to t to get

$$2c\frac{dc}{dt} = -2ab(-\sin\gamma)\frac{d\gamma}{dt}$$

so

$$\frac{dc}{dt} = \frac{ab}{c}\sin\gamma\frac{d\gamma}{dt}$$

and at 2 o'clock

$$\frac{dc}{dt} = \frac{6 \cdot 4}{\sqrt{28}}\sin\left(\frac{\pi}{3}\right)\left(-\frac{2\pi}{60} \cdot \frac{11}{12}\right)$$

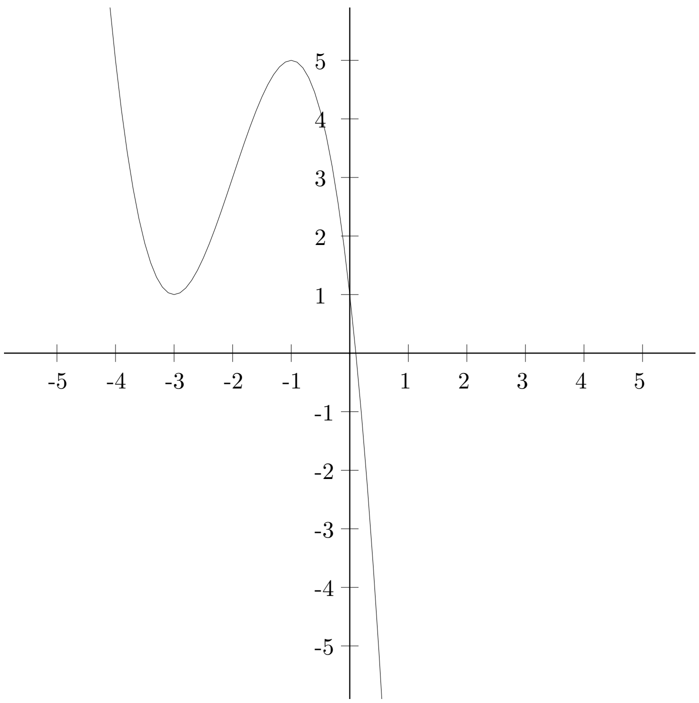
$$\frac{dc}{dt} = -\frac{24}{\sqrt{28}} \cdot \frac{\sqrt{3}}{2} \cdot \frac{2\pi}{60} \cdot \frac{11}{12}$$

Simplify to taste. As a sanity check, note that we got a negative number for the answer which means that the distance between the two hands is getting smaller. We expect this since in a few minutes, the minute hand will pass the hour hand.

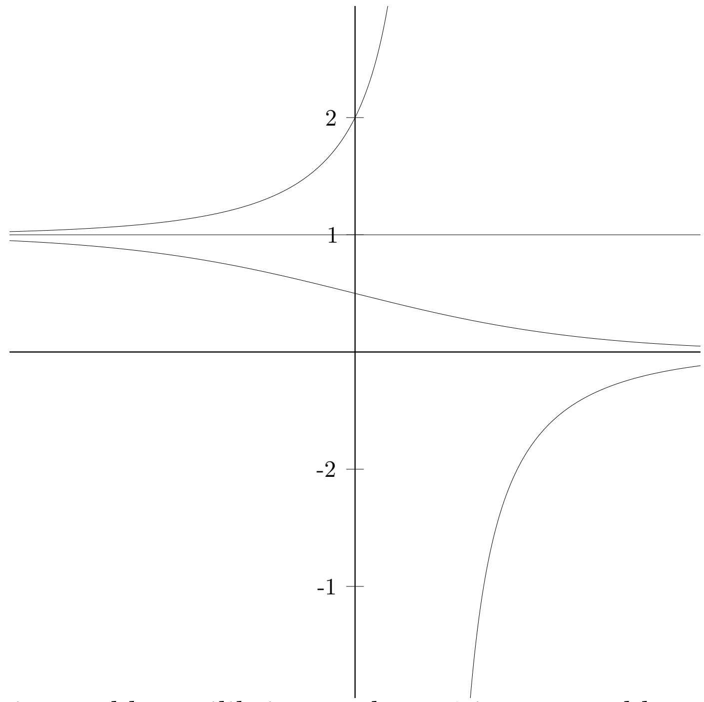
11. For $f(x) = 1 - 9x - 6x^2 - x^3$, $f'(x) = -9 - 12x - 3x^2$ and $f''(x) = -12 - 6x$. The questions asked can be answered after we have analyzed where f' and f'' are positive and where they are negative.

Start with the easiest, $f''(x) = -12 - 6x$: it vanishes only at $x = -2$. Since $f''(-3) = 6 > 0$, for $x < -2$ f'' is positive. Since $f''(-1) = -6 < 0$, for $x > -2$ f'' is negative. Hence f is concave up on $(-\infty, -2)$ and concave down on $(-2, \infty)$.

Next find where first derivative $f'(x) = -9 - 12x - 3x^2$ vanishes: $0 = -9 - 12x - 3x^2$ or $3x^2 + 12x + 9 = 0$ or $x^2 + 4x + 3 = 0$ or $(x + 1)(x + 3) = 0$, so $x = -1$ and $x = -3$ are the solutions. Evaluate $f'(-4) = -9$, $f'(-2) = 3$ and $f'(0) = -9$. Hence f is decreasing on $(-\infty, -3)$, increasing on $(-3, -1)$ and decreasing on $(-1, \infty)$.



12. The relevant questions here are: where is $\frac{dy}{dx}$ positive and where is it negative? Since $\frac{dy}{dx} = y^2 - y$, regarding the left–hand side, we see that the relevant regions are divided when $y^2 - y = 0$ or $y = 0$ and $y = 1$. In other words, the phase lines are $y = 0$ and $y = 1$. For $y > 1$ the solution curves are increasing; for y between 0 and 1 they are decreasing; and for $y < 0$ they are increasing again.



From the picture it is clear that $y = 0$ is a stable equilibrium and $y = 1$ is an unstable one.

13. Let x denote the distance of the fence facing the road, which corresponds to one side of the rectangle, and y the length of the other side. The cost of fencing the garden with these dimensions is: $C = 6x + 2y + 2x + 2y = 8x + 4y$. There are \$320 available for the construction, so: $320 = 8x + 4y$. The area is given by $A = xy$. Solving for y in the first equation, $y = 80 - 2x$. Finally: $A(x) = x(80 - 2x) = 80x - 2x^2$. Taking derivative and setting it equal to zero, $A'(x) = 80 - 4x = 0$, so $x = 20$, $y = 40$.