Exam I
September 25, 2001

11. To find the tangent line we need to know the slope, which we find via implicit differentiation:

$$(*) \qquad 2x + y' = 3\cos(x+y)(1+y')$$

so at the point $(1,-1)$ we get the equation $2 \cdot 1 + y'|_{(1,-1)} = 3\cos(0)(1+y'|_{(1,-1)}) = 3(1+y'|_{(1,-1)})$, so $y'|_{(1,-1)} = -\frac{1}{2}$. Hence an equation for the tangent line is $y-(-1) = -\frac{1}{2}(x-1)$.

The curve lies above the tangent line in a neighborhood of the point $(1,-1)$ provided $y''|_{(1,-1)} > 0$. Likewise it lies below the tangent line if $y''|_{(1,-1)} < 0$. We compute implicitly by differentiating $(*)$:

$$2 + y'' = -3\sin(x+y)(1+y')^2 + 3\cos(x+y)(y'') \ .$$

At the point $(1,-1)$ we get $2 + y''|_{(1,-1)} = -3\sin(0)(1+y')^2 + 3\cos(0)(y''|_{(1,-1)}) = 3(y''|_{(1,-1)})$, so $y''|_{(1,-1)} = 1 > 0$ and the curve lies above the tangent line in a neighborhood of the point.

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12. You are being asked to find $\frac{dV}{dt}$ at a certain moment in time. You are given that $V = \frac{\pi}{3}r^2h$ so $\frac{dV}{dt} = \frac{\pi}{3}2r\frac{dr}{dt}h + \frac{\pi}{3}r^2\frac{dh}{dt}$. The time being asked for is a time when $h = r = 2$, $\frac{dr}{dt} = 4$ and $\frac{dh}{dt} = 2$, so $\frac{dV}{dt} = \frac{\pi}{3}2 \cdot 2 \cdot 4 \cdot 2 + \frac{\pi}{3}2^2 \cdot 2 = (32+8)\frac{\pi}{3} = \frac{40}{3}\pi$.

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13. By the Mean Value Theorem, $\sin(a) - \sin(b) = \cos(c)(b-a)$ for some c between a and b. If we take $a = 2.674$ and $b = 2.670$ we see $b - a = 0.004$ and we know that $-1 \le \cos(c) \le 1$ and it follows that $-0.004 \le \sin(2.674) - \sin(2.670) \le 0.004$ which is equivalent to our absolute value inequality. Since $\frac{\pi}{2} < 2.670 < 2.674 < \pi$, we actually know $\frac{\pi}{2} < c < \pi$ and for any c in this range, $\cos(c) < 0$, so we see $\sin(2.674) - \sin(2.670) < 0$.

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14. First we locate the critical points: y' is defined and continuous everywhere since $x^2+1>0$ so we are never dividing by 0 or taking the square root of a negative number. Therefore $x=0$ is the only critical point. Similar reasoning shows y'' is defined and continuous everywhere and is never 0. Therefore,

a. f is increasing on the interval $(0,\infty)$ and decreasing on the interval $(-\infty,0)$.

b. f is concave up everywhere and concave down nowhere.

c. Since $\sqrt{x^2+1}$ is defined and continuous everywhere, the graph can have no vertical asymptotes.

To check the slant asymptote assertions we need to show $\lim_{x\to\infty}\sqrt{x^2+1}-x=0$ and $\lim_{x\to\infty}\sqrt{x^2+1}-(-x)=0$.

Compute $\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}(\sqrt{x^2+1}-x)\cdot\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=$
$\lim_{x\to\infty}\dfrac{(\sqrt{x^2+1})^2-x^2)}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\dfrac{1}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\dfrac{1}{x}\cdot\dfrac{1}{\sqrt{1+\frac{1}{x^2}}+1}=$
$\lim_{x\to\infty}0\cdot\dfrac{1}{\sqrt{1+0}+1}=0.$

Compute $\lim_{x\to-\infty}\sqrt{x^2+1}-(-x)=\lim_{x\to-\infty}(\sqrt{x^2+1}+x)\cdot\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}=$
$\lim_{x\to-\infty}\dfrac{(\sqrt{x^2+1})^2-x^2)}{\sqrt{x^2+1}-x}=\lim_{x\to-\infty}\dfrac{1}{\sqrt{x^2+1}-x}=\lim_{x\to-\infty}\dfrac{1}{x}\cdot\dfrac{1}{-\sqrt{1+\frac{1}{x^2}}-1}=$
$\lim_{x\to-\infty}0\cdot\dfrac{1}{-\sqrt{1+0}-1}=0.$