Exam II
October 28, 2003

11. To show there is only one root in the given interval, it suffices to show that p is strictly increasing or is strictly decreasing on this interval. By the Mean Value Theorem, this will follow if we can show $p'(x) > 0$ (strictly increasing) or $p'(x) < 0$ (strictly decreasing) on the relevant interval, $(0, 1)$. Since p is a polynomial, it is differentiable, and hence continuous, everywhere and hence differentiable on $(0, 1)$ and continuous on $[0, 1]$ so the Mean Value Theorem can be applied.

Compute $p'(x) = 5x^4 + 6x^2 + 2$. This is obviously $\geq 2 > 0$ since $x^4 \geq 0$ and $x^2 \geq 0$. Hence $p(x)$ is strictly increasing on $[0, 1]$ and so can have at most one root there.

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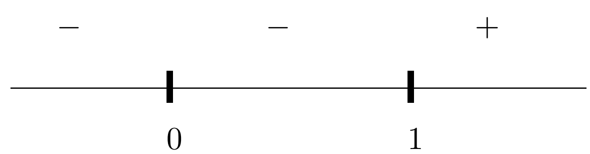
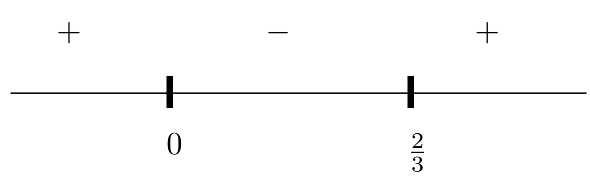
12. Begin with part b). Intervals of increase/decrease are determined by the sign of the first derivative. To determine these, locate the points where f' is 0, does not exist, or is not continuous. Since f is a polynomial, the derivative is defined and continuous everywhere, so we need to solve $f'(x) = 12x^3 - 12x^2 = 0$. The solutions are $x = 0$ and $x = 1$. Since $f'(-1) = -12 - 12 < 0$; $f'\left(\frac{1}{2}\right) = 12 \cdot \left(\frac{1}{8} - \frac{1}{4}\right) < 0$ and $f'(2) = 12 \cdot 8 - 12 \cdot 4 = 12 \cdot 4 > 0$, the signs are



Hence f is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$. Indeed both the increase and decrease are strict.

Now for part a). There is a local minimum at $x = 1$ and there are no local maxima. Since the interval $(-\infty, \infty)$ has no endpoints, there can be no extrema at the endpoints so there is no global maxima. There is a global minimum at $x = 1$ since f decreases from $-\infty$ to 1 and then increases from 1 to ∞.

For part c) we need to study $f''(x) = 36x^2 - 24x$. Since f'' is defined and continuous everywhere, the only relevant points are the solutions to $36x^2 - 24x = 0$ or $x = 0$ and $x = \frac{24}{36} = \frac{2}{3}$. Since $f''(-1) = 36 + 24 > 0$; $f''\left(\frac{1}{2}\right) = \frac{36}{4} - \frac{24}{2} = 9 - 12 < 0$ and $f''(2) = 36 \cdot 4 - 24 \cdot 2 > 0$, the signs are



Hence f is concave up on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$ and concave down on $(0, \frac{2}{3})$. Inflection points occur at $x = 0$ and at $x = \frac{2}{3}$.

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13. The statement that the oil spill forms a circular region means that its area is $A = \pi r^2$, where r is its radius. Since the area is increasing at a rate of 100 square meters per hour, we have $\frac{dA}{dt} = 100\ \mathrm{m}^2/\mathrm{hr}$. We are asked to find $\frac{dr}{dt}$ when $r = 200$ m. Well $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ by the Chain Rule, so

$$\frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{100\mathrm{m}^2/\mathrm{hr}}{2\pi\ 200\mathrm{m}} = \frac{1}{4\pi}\mathrm{m/hr}$$

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14. Answer: 12

Solution 1: $\Delta n \approx dn = n'(t)dt$, and $n'(t) = 12t$, $t = 5$, $dt = \Delta t = 5.2 - 5 = 0.2$. Thus $\Delta n \approx dn = 60(0.2) = 12$.

Solution 2: The linear approximation of $n(t)$ at $t = 5$ is $L(t) = n(5) + n'(5)(t - 5) = 350 + 60(t - 5)$. Thus $n(5.2) \approx L(5.2) = 350 + 60(0.2) = 350 + 12 = 362$. Thus $\Delta n = n(5.2) - n(5) \approx L(5.2) - L(5) = 362 - 350 = 12 = dn$.

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15. The function is continuous everywhere, hence on $[-2, 2]$. Therefore the function has an absolute maximum and an absolute minimum. The derivative is $f'(x) = -\frac{2}{3}x^{-\frac{1}{3}} = \frac{-2}{3\sqrt[3]{x}}$ which is defined everywhere except $x = 0$. Hence 0 is the only critical number. The global extrema must occur at a critical point or at an end point and since we are looking for the absolute extrema, we can proceed as follows. Calculate $f(0) = 2$; $f(-2) = 2 - (-2)^{\frac{2}{3}} = 2 - \sqrt[3]{4}$ and $f(2) = 2 - (2)^{\frac{2}{3}} = 2 - \sqrt[3]{4} = f(-2)$. Since the cube root of 4 is positive, $2 - \sqrt[3]{4} < 2$ so $2 = f(0)$ is the absolute maximum value and $2 - \sqrt[3]{4} = f(\pm 2)$ is the absolute minimum value.
