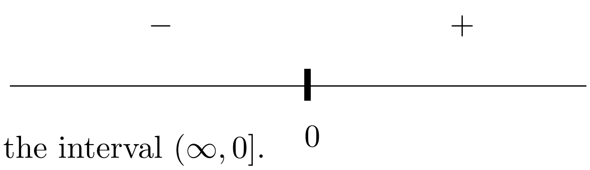
Exam III
December 2, 2003

11.

(a) To find intervals of decrease (or increase), we need to compute the derivative. The quotient rule says

$$\frac{dy}{dx} = \frac{(2x)(x^2+3) - (x^2)(2x)}{\left(x^2+3\right)^2} = \frac{2x^3+6x-2x^3}{\left(x^2+3\right)^2} = \frac{6x}{\left(x^2+3\right)^2}\ .$$

The derivative is continuous. It is defined everywhere *except* where $(x^2+3)^2 = 0$. But $(x^2+3)^2 \geq 3^2 > 0$ so the derivative is defined everywhere and vanishes precisely when $6x = 0$, or $x = 0$. Hence

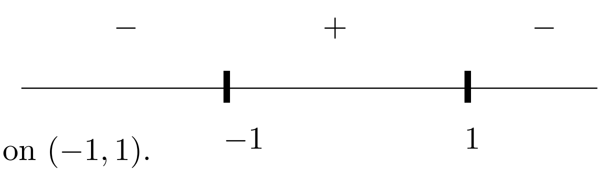


so y is decreasing on the interval $(\infty, 0]$.

(b) To compute concavity intervals, we need the second derivative:

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{d\frac{6x}{(x^2+3)^2}}{dx} = \frac{(6)\big((x^2+3)^2\big) - (6x)\big(2(x^2+3)(2x)\big)}{\big((x^2+3)^2\big)^2}$$
$$= \frac{6(x^2+3)\big((x^2+3) - (4x^2)\big)}{(x^2+3)^4} = \frac{6\big(3-3x^2\big)}{(x^2+3)^3} = \frac{18\big(1-x^2\big)}{(x^2+3)^3}\ .$$

Just as for the first derivative, the second is defined and continuous everywhere and vanishes precisely when $1 - x^2 = 0$ or $x = \pm 1$. Since $y''(0) = \frac{18}{3^3} > 0$ and $y''(\pm 2) = \frac{18(-3)}{7^3} < 0$, the relevant line is



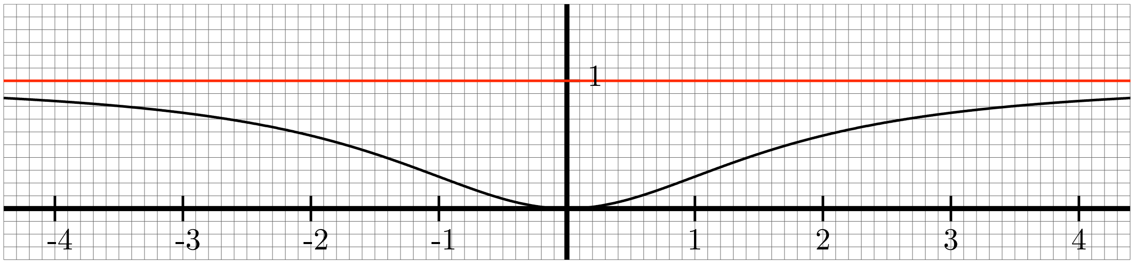
and y is concave *up* on $(-1, 1)$.

(c) *Horizontal* asymptotes are determined by $\lim\limits_{x\to\pm\infty} y$, so

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2+3} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} \cdot \frac{1}{1+\frac{3}{x^2}} = \lim_{x\to\pm\infty} \frac{1}{1+\frac{3}{x^2}} = \frac{1}{1+0} = 1\ .$$

Hence the line $y = 1$ is a horizontal asymptote both to the right and to the left. You were not asked, but because $x^2 + 3 > 0$ everywhere, there are no vertical asymptotes.

11.
(d) Here is the graph. The red line is the asymptote.



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12. The cost of the can is proportional to the area and so is the area of the bottom disk, πr^2 plus the area of the cylindrical side $2\pi rh$. Hence we need to minimize $C = \pi r^2 + 2\pi rh$. The volume of the can is $V = \pi r^2 h$ and is also 1000. Hence $h = \frac{1000}{\pi r^2}$, so $C = \pi r^2 + 2\pi r \frac{1000}{\pi r^2}$, or

$$C = \pi r^2 + \frac{2000}{r}$$

and $0 < r < \infty$. To find the critical points, compute $\frac{dC}{dr} = 2\pi r - \frac{2000}{r^2}$ which is defined and continuous on $(0, \infty)$. Hence the critical points are given by $2\pi r - \frac{2000}{r^2} = 0$, or $2\pi r^3 = 2000$, or $r^3 = \frac{1000}{\pi}$ or $r = \sqrt[3]{\frac{1000}{\pi}} = \frac{10}{\sqrt[3]{\pi}}$. The corresponding height is $h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \frac{100}{\sqrt[3]{\pi^2}}} = \frac{10}{\sqrt[3]{\pi}}$. Near 0, $\frac{dC}{dr}$ is dominated by $-\frac{1000}{r^2}$ and hence is negative; near $+\infty$, $\frac{dC}{dr}$ is dominated by $2\pi r$ and so is positive. Hence $r = \frac{10}{\sqrt[3]{\pi}}$ is a local minima. Moreover, for r from 0 to $\frac{10}{\sqrt[3]{\pi}}$, $\frac{dC}{dr}$ is decreasing and from $\frac{10}{\sqrt[3]{\pi}}$ to ∞ it is increasing, so $\frac{10}{\sqrt[3]{\pi}}$ is a global minimum.

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13. Since $v = \frac{da}{dt}$, $v(t) = \int (1 + \cos t)dt$ or $v(t) = t + \sin t + C$. Since $v(0) = 0$ and $v(0) = 0 + \sin 0 + C$, $C = 0$ and $v(t) = t + \sin t$.

Since $v = \frac{ds}{dt}$, $s(t) = \int (t + \sin t)dt$ so $s(t) = \frac{t^2}{2} - \cos t + C$. Since $s(0) = 0$ and $s(0) = \frac{0^2}{2} - \cos 0 + C$, $C = \cos 0 = 1$ and $s(t) = \frac{t^2}{2} - \cos t + 1$.

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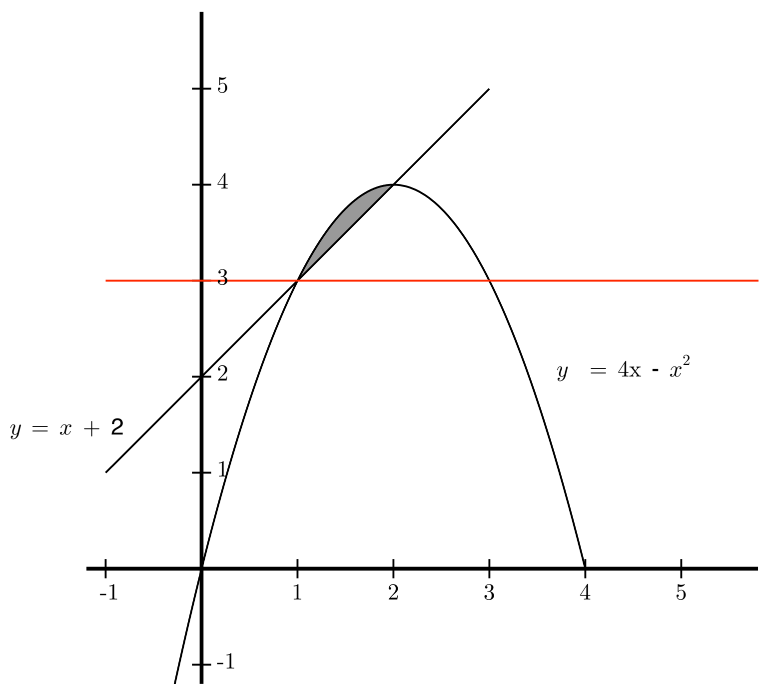
14. First put the equation into the form needed for Newton's method: solve $f(x) = 0$, where $f(x) = 2 - \sec x - \tan x$. We need $f'(x) = -\sec x \tan x - \sec^2 x$ and then

$$x_{n+1} = x_n - \frac{2 - \sec(x_n) - \tan(x_n)}{-\sec(x_n)\tan(x_n) - \sec^2(x_n)} = x_n + \frac{2 - \big(\sec(x_n) + \tan(x_n)\big)}{\sec(x_n)\big(\sec(x_n) + \tan(x_n)\big)}$$

Since $x_1 = \frac{\pi}{4}$, $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and hence $\sec(x_1) = \sqrt{2}$ and $\tan(x_1) = 1$. Therefore

$$x_2 = \frac{\pi}{4} - \frac{2 - \left((1 + \frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}(1 + \sqrt{2})} = \frac{\pi}{4} - \frac{2 - \sqrt{2}}{2 + \sqrt{2}}\,.$$

15.



Above is a graph with the shaded region the region to be rotated. The red line is the axis of revolution.

The intersection points are $(1, 3)$ and $(2, 4)$. To get washers, we must integrate with respect to x so the setup is

$$\text{Volume} = \pi \int_1^2 \left((4x - x^2) - 3\right)^2 - \left((x + 2) - 3\right)^2 \, dx$$

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15. To do the integral, write the integrand as a polynomial.

$$\big((x+2)-3\big)^2 = (x-1)^2 = x^2 - 2x + 1$$

and

$$\big((4x-x^2)-3\big)^2 = (-3+4x-x^2)^2 = (x^2-4x+3)^2$$

$$\begin{array}{ccccccccc}
 & & & & x^2 & - & 4x & + & 3 \\
 & & & & x^2 & - & 4x & + & 3 \\
\hline
x^4 & - & 4x^3 & + & 3x^2 & & & & \\
 & - & 4x^3 & + & 16x^2 & - & 12x & & \\
 & & & & 3x^2 & - & 12x & + & 9 \\
\hline
x^4 & - & 8x^3 & + & 22x^2 & - & 24x & + & 9
\end{array}$$

so

$$\begin{aligned}
\text{Volume} &= \pi \int_1^2 (x^4 - 8x^3 + 21x^2 - 22x + 8)\; dx \\
&= \pi \left(\frac{x^5}{5} - \frac{8x^4}{4} + \frac{21x^3}{3} - \frac{22x^2}{2} + 8x \Big]_1^2 \right) \\
&= \pi \left(\frac{x^5}{5} - 2x^4 + 7x^3 - 11x^2 + 8x \Big]_1^2 \right) \\
&= \pi \left(\left(\frac{32}{5} - 32 + 56 - 44 + 16 \right) - \left(\frac{1}{5} - 2 + 7 - 11 + 8 \right) \right) \\
&= \pi \left(\left(\frac{32}{5} - 4 \right) - \left(\frac{1}{5} + 2 \right) \right) \\
&= \pi \left(\frac{31}{5} - 6 \right) = \frac{\pi}{5}
\end{aligned}$$