Math. 125 Quiz #1 September 3, 2002

If

$$(f \circ g)(x) = f\big(g(x)\big) = (\sin x)^2 + 3\sin x + 7$$

what is a possibility for f and g?
If $h(u) = \cos u$ and $m(s) = 2s + 3$ what is $(h \circ m)(y) = h\big(m(y)\big)$?
If the domain of f is the interval $(3, 7)$, what is the domain of $f(x + 2)$?

. . . .

$g(x) = \sin x$ and $f(x) = x^2 + 3x + 7$.
$(h \circ m)(y) = h(m(y)) = h(2y + 3) = \cos(2y + 3)$.
The domain of $f(x + 2)$ is in the interval $(1, 5)$.

Math. 125 Quiz #2 September 10, 2002

1. What is $\displaystyle\lim_{x\to 3} \frac{3x-1}{1-x}$?

2. What is $\displaystyle\lim_{x\to 2^-} \frac{1}{2-x}$?

3. Suppose $\displaystyle\lim_{x\to 1} f(x) = 4$ and $\displaystyle\lim_{x\to 1} g(x) = -2$. What is the value of the following limit?

$$\lim_{x\to 1} \frac{f(x)\cdot g(x)}{\big(4 + g(x)\big)^2}$$

. . . .

1. $\displaystyle\lim_{x\to 3} \frac{3x-1}{1-x} = \frac{3\cdot 3-1}{1-3} = \frac{8}{-2} = -4$

2. $\displaystyle\lim_{x\to 2^-} \frac{1}{2-x} = +\infty$

3.

$$\lim_{x\to 1} \frac{f(x)\cdot g(x)}{\big(4 + g(x)\big)^2} = \frac{\lim_{x\to 1} f(x)\cdot \lim_{x\to 1} g(x)}{\big(4 + \lim_{x\to 1} g(x)\big)^2} = \frac{4\cdot(-2)}{(4-2)^2} = -2$$

Math. 125 Quiz #3 September 17, 2002

For each of the functions below, select one of the properties A-D.

A. Is continuous for all x.
B. Has one or more vertical asymptotes.
C. Has one or more removable singularities.
D. None of the above.

1. The function $[[x]] + 1$.

2. The function $\dfrac{x^2 + 3x + 2}{x + 1}$.

3. The function $\frac{(x-3)\sin x}{x^2-6x+9}$.

4. The function $x + \sqrt[3]{1-x}$.

. . . .

1. The function $[[x]] + 1$.
Answer: D
There is a jump discontinuity at each integer value.

2. The function $\dfrac{x^2 + 3x + 2}{x + 1}$.
Answer: C

$$\frac{x^2 + 3x + 2}{x + 1} = \frac{(x + 2)(x + 1)}{x + 1} = x + 2$$

Therefore, we have a removable singularity at $x = -1$.

3. The function $\frac{(x-3)\sin x}{x^2-6x+9}$.
Answer: B

$$\frac{(x - 3)\sin x}{x^2 - 6x + 9} = \frac{(x - 3)\sin x}{(x - 3)(x - 3)} = \frac{\sin x}{x - 3}.$$

Therefore, we have a vertical asymptote at $x = 3$.

4. The function $x + \sqrt[3]{1 - x}$.
Answer: A
Root functions are continuous everywhere in their domain. So $\sqrt[3]{1 - x}$ is continuous on $\mathbf{R}$. $g(x) = x$ is continuous on $\mathbf{R}$. Hence, $x + \sqrt[3]{1 - x}$ is continuous for all x.

Math. 125 Quiz #4 October 1, 2002

Find a formula for the derivative of $f(x) = \dfrac{\tan(x) + \sin(x)}{x^2 + 4x - 3}$.

. . . .

We have a quotient so the answer is
(derivative of the numerator) × (denominator) - (numerator) × (derivative of the denominator) divided by (the denominator) squared

$$f'(x) = \frac{(\sec^2(x) + \cos(x))(x^2 + 4x - 3) - (\tan(x) + \sin(x))(2x + 4)}{(x^2 + 4x - 3)^2}.$$

Math. 125 Quiz #5 October 8, 2002

1.a) Find a formula for $\dfrac{dy}{dx}$ if

$$x^4 + 5xy^2 + y^5 = -5 \ .$$

1.b) Find an equation for the tangent line to the curve at the point $(-1, -1)$.

2) Suppose that, when $x = 0$, $f'(0) = -0.5253$. If $h(x) = f(x^2 + 10x)$, find $h'(-10)$.

. . . .

1a) Using implicit differentiation:

$$4x^3 + \left(5y^2 + 10xyy'\right) + 5y^4y' = 0$$

or

$$y' = -\frac{4x^3 + 5y^2}{10xy + 5y^4}$$

1b) Line goes thru $(-1, -1)$ and has slope $\dfrac{4(-1)^3 + 5(-1)^2}{10(-1)(-1) + 5(-1)^4} = -\dfrac{5-4}{10+5} = -\dfrac{1}{15}$, so an equation for the tangent line is $y - (-1) = -\frac{1}{15}\big((x - (-1)\big)$, or $15y + 15 = -(x + 1)$.

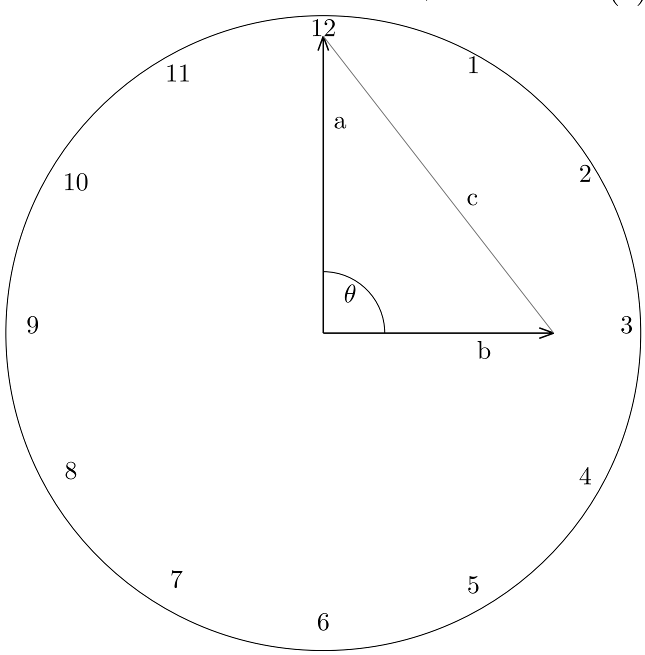
2) By the Chain Rule, $h'(x) = f'(x^2 + 10x)(2x + 10)$, so
$h'(-10) = f'(0) \cdot (2(-10) + 10) = -0.5253 \cdot (-10) = 5.253$.

Math. 125 Quiz #6 October 15, 2002

Consider a clock whose hour hand has length a and whose minute hand has length b. At 3 o'clock, how fast is the distance between the tip of the hour hand and the tip of the minute hand changing?

Let θ be the angle between the minute and hour hand. The distance, c, between the tips is given by the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)\ .$$



Step 1. It is true that $\dfrac{d\theta}{dt}$ is a constant. What constant is it?

Hint: $\theta = \theta_{\text{hr}} - \theta_{\text{min}}$ where θ_{hr} is the angle the hour hand makes with the vertical and θ_{min} is the angle the minute hand makes with the vertical. Both $\frac{d\theta_{\text{hr}}}{dt}$ and $\frac{d\theta_{\text{min}}}{dt}$ are constants. What are they?

Step 2. Use this fact and your related rates techniques to answer the original question. If the distance is *decreasing* be sure your formula gives a negative value; if increasing, a positive one.

. . . .

$$c^2 = a^2 + b^2 - 2ab\cos(\theta) \ \Rightarrow 2c\frac{dc}{dt} = -2ab(-\sin(\theta))\frac{d\theta}{dt} = 2ab\sin(\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dc}{dt} = \frac{2ab\sin(\theta)\frac{d\theta}{dt}}{2c} = \frac{ab\sin(\theta)(-\frac{11\pi}{6})}{c}$$

At 3:00, $\theta = \frac{\pi}{2}$ and $\sin(\theta) = \sin(\frac{\pi}{2}) = 1$. Therefore, at 3:00,

$$\frac{dc}{dt} = \frac{ab\sin(\frac{\pi}{2})(-\frac{11\pi}{6})}{c} = \frac{ab(-\frac{11\pi}{6})}{c} = -\frac{11ab\pi}{6c}.$$

Math. 125 Quiz #7 November 5, 2002

1. Find all the critical points for the function below. At which critical points is the function value a local maximum? Why? At which critical points is the function value a local minimum? Why?
2. The existence theorem for global max/min does not apply to this function restricted to the interval $[0,\infty)$ since the interval must be a closed interval for the theorem to apply. Nevertheless, this function does assume a minimum value on this interval. Where is it and why? Why does this function assume no maximum value on this interval?

$$\frac{(x-3)^6}{(x+1)^2}$$

. . . .

1. Let $r(x) = \frac{(x-3)^6}{(x+1)^2}$. Therefore,

$$r'(x) = \frac{(x+1)^2 \cdot 6 \cdot (x-3)^5 - (x-3)^6 \cdot 2 \cdot (x+1)}{(x+1)^4}$$

$$= \frac{(x+1)(x-3)^5[6(x+1) - 2(x-3)]}{(x+1)^4} = \frac{(x-3)^5(4x+12)}{(x+1)^3} = \frac{4(x-3)^5(x+3)}{(x+1)^3}.$$

$r'(x) = 0 \Leftrightarrow (x-3)^5(x+3) = 0 \Leftrightarrow x = -3, 3$. $r'(x)$ does not exist $\Leftrightarrow (x+1)^3 = 0 \Leftrightarrow x = -1$. So the critical numbers of $r(x)$ are $x = -3, -1, 3$. We have $x \in (-\infty, -3) \Rightarrow r'(x) < 0$, $x \in (-3, -1) \Rightarrow r'(x) > 0$, $x \in (-1, 3) \Rightarrow r'(x) < 0$ and $x \in (3, \infty) \Rightarrow r'(x) > 0$. Hence, local minima occur at $x = -3, 3$ because the derivative changes from negative to positive at these points. There is no local minimum or maximum at $x = -1$ since $r(x)$ is not defined there.

2. $\lim_{x\to\infty} r(x) = \infty$ and $r(0) = 729$. We *suspect* that the global minimum occurs at $x = 3$ since this point is in the interval of interest and a local minimum occurs here. By the first derivative test, the value of $r(x)$ is increasing on $(3, \infty)$ so $r(3) \leq r(x)$ for $x > 3$. Likewise, the value of $r(x)$ is deceasing on $[0, 3)$ so $r(x) \geq r(3)$ for x less than 3 and greater than or equal to 0. Hence $r(3) \leq r(x)$ for all $x \in [0, \infty)$, which says that the value at $x = 3$ is a global minimum for the function on the interval $[0, \infty)$.

The idea in the last paragraph is a general result. Suppose

a) a function $f(x)$ is differentiable on an interval (closed, open, or half open),
b) $x = c$ is the *only* critical number in the interval, and
c) the value of f is a local maximum (respectively minimum) at $x = c$.

Then $f(c)$ is a global maximum (respectively minimum) on that interval.

No maximum value occurs in $[0, \infty)$ since $\lim_{x\to\infty} r(x) = \infty$.

Math. 125 Quiz #8 November 12, 2002

We wish to use Newton's method to find a solution to the equation

$$\cot x = 2 \tan x \ .$$

If we make a first approximation of $x = \frac{\pi}{4}$, what does Newton's method give for a second approximation?

Leave your answer in terms of sums of fractions and rational multiples of π.

. . . .

To apply Newton's method, observe that our original equation is equivalent to $f(x) = 0$ where $f(x) = \cot x - 2 \tan x$.

Since $f'(x) = -\csc^2 x - 2 \sec^2 x$, we get

$$x_{n+1} = x_n - \frac{\cot x - 2 \tan x}{-\csc^2 x - 2 \sec^2 x} = x_n + \frac{\cot x - 2 \tan x}{\csc^2 x + 2 \sec^2 x} \ .$$

Taking $x_0 = \frac{\pi}{4}$, we get

$$x_1 = \frac{\pi}{4} - \frac{f(\frac{\pi}{4})}{f'(\frac{\pi}{4})}$$

$f(\frac{\pi}{4}) = \cot(\frac{\pi}{4}) - 2 \tan(\frac{\pi}{4}) = 1 - 2 = -1$, $f'(\frac{\pi}{4}) = -\csc^2(\frac{\pi}{4}) - 2 \sec^2(\frac{\pi}{4}) = -2 - 2 \cdot 2 = -6$ so $x_1 = \frac{\pi}{4} - \frac{-1}{-6} = \frac{\pi}{4} - \frac{1}{6}$.

Math. 125 Quiz #9: Solutions November 19, 2002

1. Consider the region in the plane below the graph $y = \csc x$; above the x–axis and between the vertical lines $x = 0$ and $x = 2$.

a. Divide the interval into 6 equal parts and write down the Riemann sum using the mid–point rule. Do not evaluate the sum or even try to simplify it.

. . . .

Midpoints are $\frac{13}{12}, \frac{15}{12}, \ldots \frac{23}{12}$: The sum is $\sum_{i=1}^{6} \frac{1}{6} \csc\left(\frac{13}{12} + \frac{i-1}{6}\right) = \sum_{i=1}^{6} \frac{1}{6} \csc\left(\frac{11}{12} + \frac{i}{6}\right) = \frac{1}{6}\left(\csc\left(\frac{13}{12}\right) + \csc\left(\frac{15}{12}\right) + \ldots + \csc\left(\frac{23}{12}\right)\right)$.

. . . .

b. Write a definite integral which evaluates the area precisely. Do not evaluate the integral.

. . . .

$\int_1^2 \csc x dx$.

. . . .

2. What number is represented by the sum

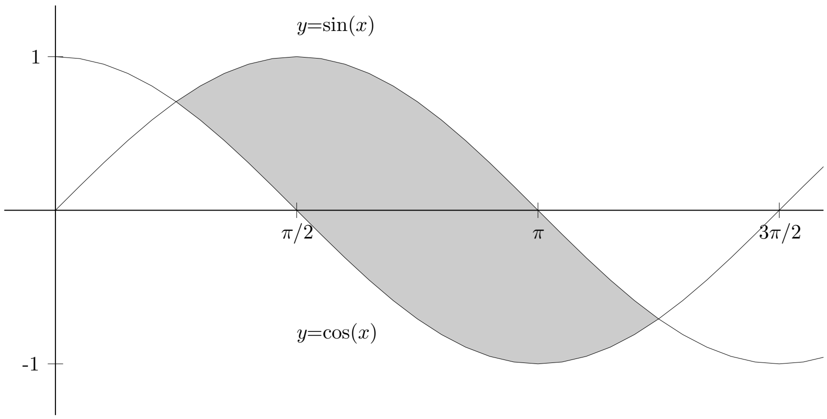
$$\sum_{q=3}^{6} (q + 2) \ ?$$

. . . .

$\sum_{q=3}^{6}(q + 2) = (3 + 2) + (4 + 2) + (5 + 2) + (6 + 2) = 26$.

Math. 125 Quiz #10 Decsmber 10, 2002

1. Write a definite integral that evaluates to the area of the shaded region below. Do not evaluate the integral.
2. Rotate the shaded region about the line $y = 2$ and write a definite integral that evaluates to the volume of the resulting solid of revolution. Do not evaluate the integral.
3. Rotate the shaded region about the line $x = -3$ and write a definite integral that evaluates to the volume of the resulting solid of revolution. Do not evaluate the integral.



. . . .

Because the curves are given as y equals a function of x, the answers will be integrals with respect to dx and so the first issue is to find the points of intersection of the two curves, or solve $\sin x = \cos x$ or $\tan x = 1$. $x = \frac{\pi}{4}$ is one solution and $x = \frac{5\pi}{4}$ is the next and these are all the solutions in the relevant interval.

The area is given by the integral $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \Big(\sin x - \cos x\Big)\ dx$: top curve minus bottom curve.

The volume integral when rotated about $y = 2$ is given by

$$\pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \Big((2 - \cos x)^2 - (2 - \sin x)^2\Big)\ dx\ .$$

With this set up, we *must* use the washer method. The outer radius is the distance between $y = 2$ and $y = \cos x$ because from the graphs, the distance from $y = 2$ to $y = \sin x$ is smaller.

The volume integral when rotated about $x = -3$ is given by

$$2\pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \Big(x - (-2)\Big) \cdot \Big(\sin x - \cos x\Big)\ dx\ .$$

With this set up, we *must* use the shell method.
