Math. 125 Quiz #1 September 2, 2003

If

$$(f \circ g)(x) = f\big(g(x)\big) = (\tan x)^3 + 3(\tan x)^2 - 3$$

find one possibility for f and g such that neither f nor g is the identity function.
If $h(u) = \cos(u^2)$ and $m(s) = 4s - 3$ write a formula for $(h \circ m)(y) = h\big(m(y)\big)$?
Solve $\frac{x^2-5x+6}{x^2-2x+1} = 0$.

. . . .

$g(x) = \tan x$ and $f(x) = x^3 + 3x^2 - 1$ is one possibility.
$(h \circ m)(y) = h\big(m(y)\big) = h(4y - 3) = \cos\big((4y - 3)^2\big)$.
For the fraction to vanish, the top (numerator) must vanish so we need to solve $x^2-5x+6 = 0$. But $x^2 - 5x + 6 = (x - 3)(x - 2)$ so the solutions are $x = 2$ and $x = 3$.

Math. 125 Quiz #2 September 9, 2003

1. What is $\displaystyle\lim_{x\to 3} \frac{x-3}{1-x}$?

2. What is $\displaystyle\lim_{x\to -1^-} \frac{x+1}{x^2-1}$?

3. Suppose $\displaystyle\lim_{x\to 1} f(x) = 3$ and $\displaystyle\lim_{x\to 1} g(x) = -2$. What is the value of the following limit?

$$\lim_{x\to 1} \frac{f(x)\cdot g(x)}{\big(2 + g(x)\big)^2}$$

. . . .

1. As $x \to 3$ $x - 3 \to 0$ and $1 - x \to -2$ so the quotient rule for limits says the answer is $\frac{0}{-2} = 0$.

2. This time, as $x \to -1^-$ $x + 1 \to 0$ and $x^2 - 1 \to 0$ so the quotient rule does not apply. But $\frac{x+1}{x^2-1} = \frac{1}{x-1}$ so $\displaystyle\lim_{x\to -1^-} \frac{x+1}{x^2-1} = \lim_{x\to -1^-} \frac{1}{x-1}$ and this is $\frac{1}{-2} = -\frac{1}{2}$ by the quotient rule.

3. This time, the top (numerator) goes to -6 and the bottom (denominator) goes to 0. It follows that there are three choices: it could be $+\infty$, it could be $-\infty$ or it could not exist for some other reason. The key here is that the denominator is a square and so is non–negative. It follows that the answer is $-\infty$.

It is true that this means that the limit requested does not exist, but the statement that it is $-\infty$ is stronger: we are saying WHY it does not exist.

Math. 125 Quiz #3 September 16, 2003

1. Find all values of c so that the function defined by

$$f(x) = \begin{cases} x^3 + 2x^2 + 4x + 3 & \text{for } x > 1 \\ x^3 - 2x^2 + cx - 3 & \text{for } x \le 1 \end{cases}$$

is continuous for all real numbers x.

2. Suppose $f(x)$ is defined and continuous for all real numbers. Further suppose that

$$\lim_{x\to 0} \frac{f(x)}{x} = 3\ .$$

Argue that $f'(0)$ exists and compute it.

. . . .

1. $f(x)$ is continuous on $(-\infty, 1)$ and on $(1, \infty)$ since polynomials are continuous.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^3 + 2x^2 + 4x + 3 = 1 + 2 + 4 + 3 = 10$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 - 2x^2 + cx - 3 = 1 - 2 + c - 3 = c - 4$$

So f is continuous at $x = 1$ if and only if $c - 4 = 10$, i.e. $c = 14$. Thus f is continuous on $(-\infty, \infty)$ if and only if $c = 14$.

2. Since f is continuous at $x = 0$, $f(0) = \lim_{x\to 0} f(x)$. On the other hand, $\lim_{x\to 0} x$ and $\lim_{x\to 0} \frac{f(x)}{x}$ exist, so we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{f(x)}{x} = 0 \cdot 3 = 0.$$

Therefore, $f(0) = \lim_{x\to 0} f(x) = 0$. So we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 3.$$

Math. 125 Quiz #4 September 30, 2003

The following questions refer to the function

$$f(x) = \frac{6\sec(x^2 - 1)}{x^2 + 4x - 3}$$

1. Find a formula for the derivative of $f(x)$.

2. Note $f(1) = 3$ and write down an equation for the tangent line to the graph of $y = f(x)$ at the point $(1, 3)$.

3. What is the y intercept of this tangent line?

. . . .

1. To use the quotient rule compute the derivative of the numerator (top)

$$6\big(\sec(x^2 - 1)\big)\big(\tan(x^2 - 1)\big)\big(2x\big) = 12x\big(\sec(x^2 - 1)\big)\big(\tan(x^2 - 1)\big)$$

and the derivative of the denominator (bottom)

$$2x + 4$$

and so

$$f'(x) = \frac{\Big(12x\big(\sec(x^2-1)\big)\big(\tan(x^2-1)\big)\Big)\Big(x^2+4x-3\Big) - \Big(6\sec(x^2-1)\Big)\Big(2x+4\Big)}{\Big(x^2+4x-3\Big)^2}$$

2. We need the slope of the tangent line at $x = 1$, which is $f'(1)$. Plugging into the formula from part 1,

$$f'(1) = \frac{-6 \cdot 4}{(1+4-3)^2} = -6$$

since $\tan(1^2 - 1) = \tan(0) = 0$. An equation for the tangent line is then

$$y - 3 = (-6)(x - 1)$$

3. To find the y–intercept, plug in $x = 0$ into the equation for the line to get $y - 3 = (-6)(-1)$ or $y = 9$. In other words, another equation for the tangent line is $y = -6x + 9$.

Math. 125 Quiz #5 October 7, 2003

1.a) Find a formula for $\frac{dy}{dx}$ if

$$x^3 + 7xy^2 + y^5 = -7\ .$$

1.b) Find an equation for the tangent line to the curve at the point $(-1, 1)$.

2) Suppose that, when $x = 0$, $f'(0) = -2$. If $h(x) = f(x^3 - 4x)$, find $h'(-2)$.

. . . .

1.a) Using implicit differentiation

$$3x^2 + \Big(7y^2 + 14xyy'\Big) + 5y^4y' = 0$$

and solve for y':

$$y' = \frac{-3x^2 - 7y^2}{14xy + 5y^4}\ .$$

1.b) The slope is

$$y'(-1,1) = \frac{-3(-1)^2 - 7(1)^2}{14(-1)(1) + 5(1)^4} = \frac{-3-7}{-14+5} = \frac{-10}{-9} = \frac{10}{9}\ .$$

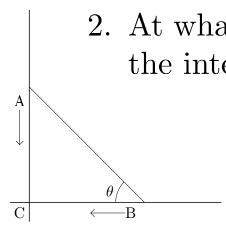
Hence the line is

$$y - 1 = \frac{10}{9}\big(x - (-1)\big)$$

Math. 125 Quiz #6 October 15, 2002

1. Car A and car B are approaching the intersection "C" of two streets intersecting at a right angle. Car A is going South at 30 mph, car B is heading West at 45 mph. We denote the angle $\angle(C, B, A)$ by θ, the distance from C to B by x, and the distance from C to A by y. Then, $\tan\theta = \frac{y}{x}$.

1. What is the angle θ when car A and car B are both 1 mile from the intersection?
2. At what rate is the angle θ changing when car A and car B are both 1 mile from the intersection?



. . . .

1. Since $x = y = 1$, $\tan\theta = 1$ so $\theta = \frac{\pi}{4}$ (or $45°$).

2. Differentiating with respect to time t,

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - y\frac{dx}{dt}}{x^2}\ .$$

At our special moment, $x = y = 1$; $\theta = \frac{\pi}{4}$ so $\sec\theta = \sqrt{2}$; $\frac{dy}{dt} = -30$ and $\frac{dx}{dt} = -45$. Hence

$$2\frac{d\theta}{dt} = \frac{(-30)\cdot 1 - 1\cdot(-45)}{1^2} = -30 + 45 = 15$$

so $\frac{d\theta}{dt} = \frac{15}{2}$. The units are radians per hour.

Math. 125 Quiz #7 November 4, 2003

1. Find all the critical points for the function below. At which critical points is the function value a local maximum? Why? At which critical points is the function value a local minimum? Why?
2. The existence theorem for global max/min does not apply to this function restricted to the interval $(1, \infty)$ since the interval must be a closed interval for the theorem to

apply. Nevertheless, this function does assume an ABSOLUTE MINIMUM value on this interval. Where is it and why? Why does this function assume no ABSOLUTE MAXIMUM on this interval?

$$\frac{(x-2)^4}{(x-1)^6}$$

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1. The critical points are the points *in the domain of the function* for which $f' = 0$ or where f' does not exist.

$$f'(x) = \frac{4(x-2)^3(x-1)^6 - (x-2)^4 6(x-1)^5}{\left((x-1)^6\right)^2} = (x-2)^3(x-1)^5\left(\frac{4(x-1)-6(x-2)}{(x-1)^{12}}\right)$$
$$= \frac{(x-2)^3}{(x-1)^7}\left(8-2x\right)$$

Now f' vanishes at $x=2$ and $x=4$ and is undefined at $x=1$. Since f is not defined at $x=1$, the critical points are $x=2$ and $x=4$. The signs of f' on the interval $(-\infty, \infty)$ are

$$\begin{array}{ccccccc} + & & - & & + & & - \\ \hline & 1 & & 2 & & 4 & \end{array}$$

It follows that there is a local minimum at $x=2$ and a local maximum at $x=4$.

2. First note that $\lim_{x\to 1^+} \frac{(x-2)^4}{(x-1)^6} = +\infty$ so f has no absolute maximum on $(1,\infty)$.

On $(1,2)$ f is decreasing so for $x \in (1,2)$, $f(x) > f(2) = 0$. On $(2,4)$ f is increasing, so for $x \in (2,4)$, $f(x) > f(2) = 0$, so $f(x) \geq 0$ for $x \in (1,4)$. On $[4,\infty)$ f is strictly decreasing, so $f(x) > \lim_{x\to\infty} f(x)$. Since $\lim_{x\to\infty} \frac{(x-2)^4}{(x-1)^6} = 0$, $f(x) > 0$ for all $x \in [4,\infty)$. Therefore $f(x) \geq 0$ on $(1,\infty)$ or equivalently, f has a global minimum on $(1,\infty)$. The minimum value is 0 and it occurs only once at $x=2$.

Math. 125 Quiz #8 November 10, 2003

We wish to use Newton's method to find a solution to the equation

$$\sin x = 2\cos x\ .$$

If we make a first approximation of $x = \frac{\pi}{3}$, what does Newton's method give for a second approximation?

Leave your answer in terms of sums of fractions, rational multiples of square roots of integers and rational multiples of π.

. . . .

The general formula is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Here $f(x) = \sin x - 2\cos x$; $f'(x) = \cos x + 2\sin x$. Since $x_0 = \frac{\pi}{3}$,

$$x_1 = \frac{\pi}{3} - \frac{\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right)}$$

Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ so

$$x_1 = \frac{\pi}{3} - \frac{\frac{\sqrt{3}}{2} - 2\cdot\frac{1}{2}}{\frac{1}{2} + 2\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - \frac{\sqrt{3}-2}{1+2\sqrt{3}}$$

Math. 125 Quiz #9 November 18, 2003

1. Consider the region in the plane below the graph $y = 0.3 + \tan x$; above the x–axis and between the vertical lines $x = 3$ and $x = 4$. The graph really is above the x–axis on this interval.

  a. Divide the interval into 5 equal parts and write down the Riemann sum using the right endpoints. Do not evaluate the sum or even try to simplify it.

  b. Write a definite integral which evaluates the area precisely. Do not evaluate the integral.

2. This question is to test your understanding of the sigma notation. What number is represented by the sum

$$\sum_{q=3}^{6}\left(\frac{1}{q} - \frac{1}{q+1}\right)\ ?$$

. . . .

1. a The endpoints of the resulting intervals are $x_0 = 3$; $x_1 = 3 + \frac{4-3}{5} = 3.2$; $x_2 = 3.4$; $x_3 = 3.6$; $x_4 = 3.8$; and $x_5 = 4$. Hence the Riemann sum is

$$f(3.2)\cdot 0.2 + f(3.4)\cdot 0.2 + f(3.6)\cdot 0.2 + f(3.8)\cdot 0.2 + f(4)\cdot 0.2$$

so

$$\begin{aligned}&\big(0.3 + \tan(3.2)\big)\cdot 0.2 + \big(0.3 + \tan(3.4)\big)\cdot 0.2 + \big(0.3 + \tan(3.6)\big)\cdot 0.2 +\\ &\big(0.3 + \tan(3.8)\big)\cdot 0.2 + \big(0.3 + \tan(4)\big)\cdot 0.2\end{aligned}$$

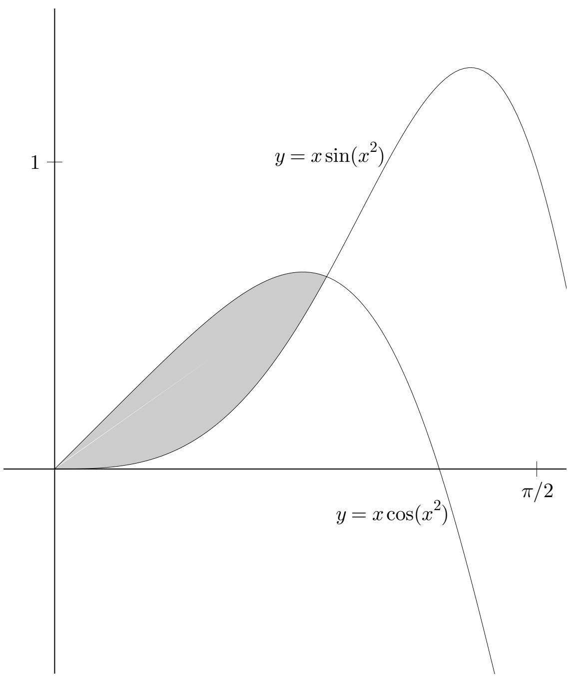
b. The actual area is $\displaystyle\int_3^4 0.3+\tan(x)\,dx$.

2.

$$\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)=\frac{1}{3}-\frac{1}{7}=\frac{7-3}{21}=\frac{4}{21}$$

Math. 125 Quiz #10 November 25, 2003

1. Write a definite integral that evaluates to the area of the shaded region below.
2. Evaluate the integral.



. . . .

Since the curves are given as y equals a function of x and since solving the equations for x as a function of y would be hard, we decide to integrate along the x–axis. Then the integral starts at $x = 0$ and continues until the x–coordinate of the intersection point, which we will have to determine. On this interval, the upper curve is $y = x\cos(x^2)$ and the lower curve is $y = x\sin(x^2)$.

The intersection points are solutions to the equation

$$x\cos(x^2) = x\sin(x^2)$$

One solution is $x = 0$. If $x \neq 0$, then $\cos(x^2) = \sin(x^2)$. To solve this, first start by solving $\cos(z) = \sin(z)$. Whenever $\cos(z) = 0$, $\sin(z) \neq 0$, so this equation is equivalent to the equation $\tan(z) = 1$ or $z = \frac{\pi}{4} + k\pi$ for k and integer. Hence $x^2 = \frac{\pi}{4} + k\pi$ or

$x = \pm\sqrt{\frac{\pi}{4} + k\pi}$. If $k < 0$, $\frac{\pi}{4} + k\pi < 0$ and we can not take the square root: for $k \geq 0$, $\frac{\pi}{4} + k\pi$ is increasing as k increases, so the intersection point on the graph is $\sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

Hence the answer to 1 is

$$\int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2) - x\sin(x^2)\ dx\ .$$

To actually do the integrals start with $\int x\cos(x^2)dx$ and try the substitution $u = x^2$: $du = 2xdx$ so $\int x\cos(x^2)dx = \int \frac{1}{2}\cos(u)du = \frac{\sin(u)}{2} + C = \frac{\sin(x^2)}{2} + C$. The second integral succumbs to the same substitution, or else guess and check: $\int x\sin(x^2)dx = \frac{-\cos(x^2)}{2} + C$ and finish by plugging in $x = \frac{\sqrt{\pi}}{2}$ and $x = 0$ and subtract.

Better yet, try the substitution $u = x^2$ on the entire integral and change limits:

$$\begin{aligned}
\int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2) - x\sin(x^2)\ dx &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos(u) - \sin(u)du \\
&= \frac{1}{2}\big(\sin(u) + \cos(u)\big)\Big]_0^{\frac{\pi}{4}} = \frac{1}{2}\Big(2\frac{\sqrt{2}}{2} - 1\Big) = \frac{\sqrt{2}}{2} - \frac{1}{2}\ .
\end{aligned}$$
