§6.3

#1. a) $e^{\ln 7.2} = 7.2$: since ($e^{\ln x} = x$).
b) $e^{-\ln\, x^2} = e^{-\ln(x^2)} = \frac{1}{e^{\ln(x^2)}} = \frac{1}{x^2} = x^{-2}$.
c) $e^{\ln x - \ln y} = \frac{e^{\ln x}}{e^{\ln y}} = \frac{x}{y}$.

#3. a) $2\ln\sqrt{e} = 2\ln(e^{1/2}) = 2 \cdot \frac{1}{2} \cdot \ln e = 2\frac{1}{2} \cdot 1 = 1$.
b) $\ln(\ln e^e) = \ln(e \ln e) = \ln(e \cdot 1) = \ln e = 1$.
c) $\ln(e^{-x^2-y^2}) = -x^2 - y^2$: (since $\ln e^z = z$).

#5. Solve $\ln y = 2t + 4$, so $y = e^{\ln y} = e^{2t+4}$.

#9. $\ln(y-1) - \ln 2 = x + \ln x$, so $\ln(y-1) = x + \ln x + \ln 2$ and $y - 1 = e^{\ln(y-1)} = e^{x+\ln x+\ln 2} = e^x \cdot e^{\ln x} \cdot e^{\ln 2} = 2xe^x$, so $y = 1 + 2xe^x$.

#13. a) Solve $e^{-0.3t} = 27$: $(-0.3)t = \ln(e^{-0.3t}) = \ln(27) = \ln(3^3) = 3\ln 3$. Hence $t = -\frac{3\ln 3}{0.3} = -10\ln 3$.
b) Solve $e^{kt} = \frac{1}{2}$: $kt = \ln(e^{kt}) = \ln(\frac{1}{2}) = -\ln 2$, so $t = -\frac{\ln 2}{k}$.
c) Solve $e^{(\ln 0.2)t} = 0.4$: $(\ln 0.2)t = \ln\left(e^{(\ln 0.2)t}\right) = \ln(0.4)$, so $t = \frac{\ln 0.4}{\ln 0.2}$.

#19. $y = e^{5-7x}$: find $\frac{dy}{dx}$. $\frac{dy}{dx} = e^{5-7x}\frac{d(5-7x)}{dx} = e^{5-7x}(-7) = -7e^{5-7x}$.

#23. $y = (x^2 - 2x + 2)e^x$: find $\frac{dy}{dx}$. $\frac{dy}{dx} = \left(\frac{d(x^2-2x+2)}{dx}\right)e^x + (x^2 - 2x + 2)\frac{d(e^x)}{dx} = (2x-2)e^x + (x^2 - 2x + 2)e^x = x^2\, e^x$.

#27. $y = \cos\left(e^{-\theta^2}\right)$: find $\frac{dy}{d\theta}$. $\frac{dy}{d\theta} = -\left(\sin\left(e^{-\theta^2}\right)\right)\frac{d\left(e^{-\theta^2}\right)}{d\theta} = -\left(\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d(-\theta^2)}{d\theta} = -\left(\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)(-2\theta) = 2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$.

#37. $\ln y = e^y \sin x$: find $\frac{dy}{dx}$. This uses implicit differentiation, or more fundamentally, the Chain Rule. Differentiate both sides with respect to x, treating y as a function of x. $\frac{y'}{y} = \frac{d(e^y)}{dx}\sin x + e^y\frac{d(\sin x)}{dx} = e^y y' \sin x + e^y \cos x$. Now solve for y' in terms of x and y: $y' = y'ye^y \sin x + ye^y \cos x$, or $y' - y'ye^y \sin x = ye^y \cos x$ or $y' = \frac{ye^y \cos x}{1 - ye^y \sin x}$.

#41. $\displaystyle\int \left(e^{3x} + 5e^{-x}\right) dx = \int e^{3x}\, dx + 5\int e^{-x}\, dx = \frac{e^{3x}}{3} + 5\frac{e^{-x}}{-1} + C = \frac{e^{3x}}{3} - 5e^{-x} + C.$

#49. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}} dr$. Substitute $t = \sqrt{r}$: $dt = \frac{1}{2}r^{-1/2}dr = \frac{1}{2\sqrt{r}}dr$, or $\frac{dr}{\sqrt{r}} = 2dt$ and our integral becomes $\displaystyle\int e^t(2dt) = 2e^t + C = 2e^{\sqrt{r}} + C$.

#55. $\displaystyle\int_0^{\frac{\pi}{4}} \left(1 + e^{\tan\theta}\right)\sec^2\theta\, d\theta$. Substitute $s = \tan\theta$: $ds = \sec^2\theta d\theta$. Hence our integral becomes $\displaystyle\int_0^1 (1 + e^s)\, ds = (s + e^s)\Big|_0^1 = (1 + e) - (0 + e^0) = 1 + e - 1 = e$.

#63. Solve $\frac{dy}{dt} = e^t \sin\big(e^t - 2\big)$, $y(\ln 2) = 0$. First integrate to find all functions satisfying the differential equation: $y = \displaystyle\int e^t \sin\big(e^t - 2\big)\, dt$: substitute $s = e^t$: $ds = e^t dt$ so $y = \displaystyle\int \sin(s-2)\, ds = -\cos(s-2) + C = -\cos(e^t - 2) + C$. Now $y(\ln 2) = -\cos(e^{\ln 2} - 2) + C = -\cos(2-2) + C = -\cos 0 + C = C - 1$. But $y(\ln 2)$ is also 0, so $C = 1$ and $y = 1 - \cos(e^t - 2)$.

#65. $\frac{d^2y}{dx^2} = 2e^{-x}$, $y(0) = 1$, $y'(0) = 0$. Integrate twice. $\frac{dy}{dx} = \displaystyle\int 2e^{-x} = 2\frac{e^{-x}}{=1} + C = -2e^{-x} + C$. When $x = 0$ $y' = 0$ and also $-2e^{-0} + C = C - 2e^0 = C - 2$. Hence $C = 2$ and $y' = 2 - 2e^{-x}$. But then $y = \displaystyle\int (2 - 2e^{-x})\, dx = 2x - (-2e^{-x}) + C = 2x + 2e^{-x} + C$. Since $y(0) = 1$, $1 = 2 \cdot 0 + 2e^{-0} + C = 2 + C$ so $C = 1$ and $y = 1 + 2x + 2e^{-x}$.

#69. Find absolute minimum value of $f(x) = x^2 \ln\big(\frac{1}{x}\big)$. Locate critical points: solve $0 = f'(x) = 2x \ln\big(\frac{1}{x}\big) + x^2\left(\frac{\frac{d(\frac{1}{x})}{dx}}{\frac{1}{x}}\right) = -2x \ln x + x^2\big(-x^{-2} \cdot x\big) = -2x \ln x - x = (-x)(1 + 2\ln x)$. This product vanishes if $x = 0$ or if $1 + 2\ln x = 0$. The point $x = 0$ is not in the domain of f so it is not a critical point. Hence the only critical point occurs when $1 + 2\ln x = 0$, or $\ln x = -\frac{1}{2}$ and $x = e^{\ln x} = e^{-1/2} = \frac{1}{\sqrt{e}}$. The value of f at $x = \frac{1}{\sqrt{e}}$ is $(\frac{1}{\sqrt{e}})^2 \ln\left(\frac{1}{\frac{1}{\sqrt{e}}}\right) = -\frac{1}{e} \cdot \ln\sqrt{e} = -\frac{1}{e} \cdot \ln\big(e^{1/2}\big) = -\frac{1}{e} \cdot (\frac{1}{2}) = -\frac{1}{2e}$. We still have to see that f has a relative maximum. Compute $f'(1) = (-1)\big(1 - 2\ln 1\big) = -(1 - 2\cdot 0) = -1 < 0$, so f is idecreasing to the right of $\frac{1}{\sqrt{e}}$ (note $1 < \sqrt{e}$). Compute $f'(e^{-2}) = -e^{-2}\big(3 + 2\ln(e^{-2})\big) = -e^{-2} \cdot (3 - 4) = e^{-2} > 0$ so f is increasing to the left of $\frac{1}{\sqrt{e}}$. As an alternative to show f has a local minimum at $\sqrt{e}$ apply the 2nd Derivative Test. $\frac{d^2 f}{dx^2} = -\frac{d\Big(x(1+2\ln x)\Big)}{dx} = -\Big(1 \cdot (1 + 2\ln x) + x \cdot (\frac{2}{x})\Big) = -(1 + 2\ln x + 2) = -(3 + 2\ln x)$. Hence $f''(\frac{1}{\sqrt{e}}) = -\big(3 + 2\ln(\frac{1}{\sqrt{e}})\big) = -(3-1) = -2 < 0$, so the 2nd Derivative Test shows f has a local maximum at $x = \frac{1}{\sqrt{e}}$. Since f has only the one critical point and is continuous, the local maximum must be an absolute (or global) maximum.

#71. Draw a graph and determine that the region is bounded on the left by the vertical line $x = 0$ (since this is where the two graphs intersect); on the left by $x = \ln 3$ (we were told this); above by $y = e^{2x}$ and below by $y = e^x$. The area is given by $\displaystyle\int_0^{\ln 3} (e^{2x} - e^x)\, dx$ and this is the formula which should be familiar to you (using calculus to find the area of planar regions §5.1).

#72. Same ideas as 71. The curves still intersect at $x = 0$, so the area is $\displaystyle\int_0^{2\ln 2} (e^{x/2} - e^{-x/2})\, dx$.

#73. The length of the graph of $y = f(x)$ between the point $\big(a, f(a)\big)$ and the point $\big(b, f(b)\big)$

is $\displaystyle\int_a^b \sqrt{1+\big(f'(x)\big)^2}\,dx$. Once you recall this formula (§5.5) and stare at the problem, you see you are being asked to find a function $f(x)$ such that $f'(x) = \sqrt{\frac{1}{4}e^x} = \frac{e^{x/2}}{2}$. $f(x) = e^{x/2}$ is such a function. Any function of the form $e^{x/2} + C$ is also such a function, so the problem has many answers.

#74. From §5.6, we recall that the surface area of the surface of revolution obtained by revolving the graph $s = f(t)$ about the t axis for $a \le t \le b$ is given by

$$2\pi \int_a^b f(t)\sqrt{1+\big(f'(t)\big)^2}\,dt\ .$$

In our case $s = x$; $y = t$; $f(t) = \frac{e^t+e^{-t}}{2}$; $a = 0$; $b = \ln 2$. $f'(t) = \frac{e^t-e^{-t}}{2}$; $\big(f'(t)\big)^2 = \frac{e^{2t}-2+e^{-2t}}{4}$ so $1+\big(f'(t)\big)^2 = \frac{e^{2t}+2+e^{-2t}}{4} = \left(\frac{e^t+e{-t}}{2}\right)^2$, so our integral is $\displaystyle\int_0^{\ln 2} \frac{e^t+e{-t}}{2}\sqrt{\left(\frac{e^t+e{-t}}{2}\right)^2}\,dt = \int_0^{\ln 2}\left(\frac{e^t+e^{-t}}{2}\right)^2 dt = \int_0^{\ln 2}\frac{e^{2t}+2+e^{-2t}}{4}dt = \cdots$.