§6.6

#7. For $\lim_{t\to 0} \frac{\sin t^2}{t}$, the form is $\frac{0}{0}$ so l'Hôpital's Rule applies. First calculate $\frac{d \sin t^2}{dt} = -(\cos t^2)\frac{dt^2}{dt} = (-2t)\cos t^2$, so our limit is $\lim_{t\to 0} \frac{(-2t)\cos t^2}{1} = \lim_{t\to 0} -(2t)\cos t^2 = 0$.

#13. Find $\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta}$. Since $\sin(\frac{\pi}{2}) = -1$ and $\cos(2\frac{\pi}{2}) = -1$, the form is $\frac{0}{0}$. Hence l'Hôpital's Rule applies and our limit is $\lim_{\theta\to\frac{\pi}{2}} \frac{-\cos\theta}{-2\sin(2\theta)}$. Since $\cos(\frac{\pi}{2}) = 0 = \sin(2\frac{\pi}{2})$, this second limit also has the form $\frac{0}{0}$ and l'Hôpital's Rule applies. $\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta}{-4\cos(2\theta)} = \frac{1}{(-4)(-1)} = \frac{1}{4}$.

#15. Find $\lim_{x\to 0} \frac{x^2}{\ln(\sec x)}$. Since $\sec 0 = 1$, $\ln(\sec 0) = 0$ so we have the form $\frac{0}{0}$ and l'Hôpital's Rule applies. To apply l'Hôpital's Rule, we need to compute $\frac{d\ln(\sec x)}{dx} = \frac{\frac{d\sec x}{dx}}{\sec x} = (\cos x)(\sec x)(\tan x) = \tan x$. Our limit is $\lim_{x\to 0} \frac{2x}{\tan x}$ which again has the form $\frac{0}{0}$. There are at least two ways to proceed. First apply l'Hôpital's Rule: $\lim_{x\to 0} \frac{2}{\sec^2 x} = \frac{2}{1} = 2$. A second way is to simplify first: $\lim_{x\to 0} \frac{2x}{\tan x} = \lim_{x\to 0} (\cos x)\frac{2x}{\sin x} = (2\cos x) \lim_{x\to 0} \frac{x}{\sin x}$ and we have calculated $\lim_{x\to 0} \frac{x}{\sin x} = 1$, so our limit is $(2\cos(0)) \cdot 1 = 2$.

#19. Find $\lim_{x\to\frac{\pi}{2}^-} \left(x-\frac{\pi}{2}\right)\sec x$. The form if $0{\cdot}\infty$ so we write $\lim_{x\to\frac{\pi}{2}^-} \left(x-\frac{\pi}{2}\right)\sec x = \lim_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos x}$ and this limit has the form $\frac{0}{0}$. By l'Hôpital's Rule, $\lim_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{-\sin x}$ and since $\sin(\frac{\pi}{2}) = 1$, our limit is -1.

#27. Find $\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\ln x}$ which has the form $\frac{\infty}{\infty}$. To apply l'Hôpital's Rule, compute $\frac{d\ln(x^2+2x)}{dx} = \frac{2x+2}{x^2+2x}$ so $\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\ln x} = \lim_{x\to 0^+} \frac{\frac{2x+2}{x^2+2x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x(2x+2)}{x^2+2x} = \lim_{x\to 0^+} \frac{2x+2}{x+2} = \frac{2}{2} = 1$.

#31. Find $\lim_{x\to\infty} \big(\ln(2x) - \ln(x+1)\big)$ which has the form $\infty - \infty$. Write $\lim_{x\to\infty} \big(\ln(2x) - \ln(x+1)\big) = \lim_{x\to\infty} \big(\ln(x+1)\big)\big(\frac{\ln(2x)}{\ln(x+1)} - 1\big)$. Compute $\lim_{x\to\infty} \frac{\ln(2x)}{\ln(x+1)}$ using l'Hôpital's Rule: $\lim_{x\to\infty} \frac{\ln(2x)}{\ln(x+1)} = \lim_{x\to\infty} \frac{\frac{2}{2x}}{\frac{1}{x+1}} = 1$ so this time we have the form $\infty \cdot 0$ and we rewrite again: $\lim_{x\to\infty} \big(\ln(2x) - \ln(x+1)\big) = \lim_{x\to\infty} \frac{\frac{\ln(2x)}{\ln(x+1)} - 1}{\frac{1}{\ln(x+1)}}$ which has the form $\frac{0}{0}$. Compute $\frac{d\frac{\ln(2x)}{\ln(x+1)}}{dx} =$

$\dfrac{\frac{d\ln(2x)}{dx}\ln(x+1)-(\ln 2x)\frac{d\ln(x+1)}{dx}}{\big(\ln(x+1)\big)^2} = \dfrac{\frac{\ln(x+1)}{x}-\frac{\ln(2x)}{x+1}}{\big(\ln(x+1)\big)^2}$; $\dfrac{d\frac{1}{\ln(x+1)}}{dx} = -\dfrac{1}{(x+1)\big(\ln(x+1)\big)^2}$ so $\lim_{x\to\infty}\big(\ln(2x)-\ln(x+1)\big) = \lim_{x\to\infty}\dfrac{\frac{\frac{\ln(x+1)}{x}-\frac{\ln(2x)}{x+1}}{\big(\ln(x+1)\big)^2}}{-\frac{1}{(x+1)\big(\ln(x+1)\big)^2}} = -\lim_{x\to\infty}\dfrac{(x+1)\ln(x+1)}{x}-\ln(2x)$. This looks so much like the original problem that one feels one is making no progress. We try a different approach. Note $e^{\lim_{x\to\infty}\big(\ln(2x)-\ln(x+1)\big)} = \lim_{x\to\infty} e^{\ln(2x)-\ln(x+1)} = \lim_{x\to\infty}\frac{2x}{x+1}$. This last limit can be computed using l'Hôpital's Rule again to get 2. Since $e^{\lim_{x\to\infty}\big(\ln(2x)-\ln(x+1)\big)} = 2$, $\lim_{x\to\infty}\big(\ln(2x)-\ln(x+1)\big) = \ln 2$.

#37. Find $\lim_{x\to\infty}\int_x^{2x}\frac{1}{t}\,dt = \lim x\to\infty\ln(2x)-\ln x$. Just as in 31, this last limit is $\ln 2$.

#42. $\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty}\frac{x^2}{e^x}$. l'Hôpital's Rule applies ($\frac{\infty}{\infty}$) to get $\lim_{x\to\infty}\frac{2x}{e^x}$ which still has form $\frac{\infty}{\infty}$, so $\lim_{x\to\infty}\frac{2}{e^x} = 0$.

#43. Find $\lim_{x\to 1^+} x^{\frac{1}{1-x}} = \lim_{x\to 1^+} e^{\frac{\ln x}{1-x}} = e^{\lim_{x\to 1^+}\frac{\ln x}{1-x}}$. We compute $\lim_{x\to 1^+}\frac{\ln x}{1-x}$ using l'Hôpital's Rule, so compute $\lim_{x\to 1^+}\frac{\frac{1}{x}}{-x} = -1$. Hence $\lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}$.

#51. Find $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}$, so we compute $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$ and now we may apply l'Hôpital's Rule. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$. Hence $\lim_{x\to 0^+} x^x = e^0 = 1$.

#53. Find $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}$. l'Hôpital's Rule applies and $\frac{d\sqrt{9x+1}}{dx} = \frac{9}{2\sqrt{9x+1}}$ and $\frac{d\sqrt{x+1}}{dx} = \frac{1}{2\sqrt{x+1}}$ so we see $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}} = \lim_{x\to\infty}\frac{\frac{9}{2\sqrt{9x+1}}}{\frac{1}{2\sqrt{x+1}}} = 9\lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{9x+1}} = \dfrac{9}{\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}}$. If $L = \lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}$ this shows $L^2 = 9$. Somewhat more easily, $L^2 = \left(\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}\right)^2 = \lim_{x\to\infty}\frac{9x+1}{x+1} = 9$. Since $\frac{\sqrt{9x+1}}{\sqrt{x+1}} > 0$ for large x, we see $L = 3$ or the limit does not exist, but we have no theorems to help us understand which of these is the case. To actually compute the limit, proceed as follows. Since $\frac{\sqrt{9x+1}}{\sqrt{x+1}} > 0$ for large x $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim_{x\to\infty}\frac{9x+1}{x+1}}\sqrt{9} = 3$.

#57. Calculation b) is correct and follows from the usual quotient rule for limits. Calculation a) is an incorrect application of l'Hôpital's Rule.

#63. Find $\lim\limits_{k\to\infty}\left(1+\dfrac{r}{k}\right)^k = \lim\limits_{k\to\infty} e^{k\ln(1+\frac{r}{k})} = e^{\lim_{k\to\infty} k\ln(1+\frac{r}{k})}$. Compute $\lim\limits_{k\to\infty} k\ln(1+\dfrac{r}{k}) = \lim\limits_{k\to\infty} \frac{\ln(1+\frac{r}{k})}{\frac{1}{k}} = \lim\limits_{k\to\infty} \frac{\frac{\frac{-r}{k^2}}{1+\frac{r}{k}}}{\frac{-1}{k^2}} = r\lim\limits_{k\to\infty}\frac{1}{1+\frac{r}{k}} = r$. Hence $\lim\limits_{k\to\infty}\left(1+\dfrac{r}{k}\right)^k = e^r$.