§6.10

#3. If $\cosh x = \frac{17}{15}$, $x > 0$ find the values of the remaining hyperbolic trig. functions. By definition $\operatorname{sech} x = \frac{15}{17}$. From $\cosh^2 x - \sinh^2 x = 1$ we see $\sinh^2 x = \left(\frac{17}{15}\right)^2 - 1 = \frac{289}{225} - 1 = \frac{64}{225}$ so $\sinh x = \frac{8}{15}$. By definition $\operatorname{csch} x = \frac{15}{8}$; $\tanh x = \frac{\frac{8}{15}}{\frac{17}{15}} = \frac{8}{17}$; and $\coth x = \frac{17}{8}$.

#15. If $y = 2\sqrt{t}\ \tanh\sqrt{t}$, find $\frac{dy}{dt}$. By the Product Rule, $\frac{dy}{dt} = \left(\frac{d2\sqrt{t}}{dt}\right)\tanh\sqrt{t} + \left(2\sqrt{t}\right)\frac{d\tanh\sqrt{t}}{dt}$. From the Power Rule, $\frac{d2\sqrt{t}}{dt} = 2\frac{1}{2}t^{-1/2} = \frac{1}{\sqrt{t}}$. From the formula for the derivative of tanh and the Chain Rule, $\frac{d\tanh\sqrt{t}}{dt} = \left(\operatorname{sech}^2(\sqrt{t})\right)\frac{d\sqrt{t}}{dt} = \left(\operatorname{sech}^2\sqrt{t}\right)\left(\frac{1}{2}t^{-1/2}\right) = \frac{\operatorname{sech}^2\sqrt{t}}{2\sqrt{t}}$. Hence $\frac{dy}{dt} = \frac{\tanh\sqrt{t}}{\sqrt{t}} + \operatorname{sech}^2\sqrt{t}$.

#19. If $y = \operatorname{sech}\theta\ (1 - \ln\operatorname{sech}\theta)$ find $\frac{dy}{d\theta}$. From the Product Rule, it follows that $\frac{dy}{d\theta} = \left(\frac{d\operatorname{sech}\theta}{d\theta}\right)\left(1 - \ln\operatorname{sech}\theta\right) + \left(\operatorname{sech}\theta\right)\left(\frac{d(1-\ln\operatorname{sech}\theta)}{d\theta}\right)$. From the book, $\frac{d\operatorname{sech}\theta}{d\theta} = -\operatorname{sech}\theta\tanh\theta$. From the Sum Rule, the Chain Rule applied to the ln, $\frac{d(1-\ln\operatorname{sech}\theta)}{d\theta} = 0 - \left(\frac{\frac{d\operatorname{sech}\theta}{d\theta}}{\operatorname{sech}\theta}\right) = -\frac{-\operatorname{sech}\theta\tanh\theta}{\operatorname{sech}\theta} = \tanh\theta$. Plugging back in, $\frac{dy}{d\theta} = \left(-\operatorname{sech}\theta\tanh\theta\right)\left(1 - \ln\operatorname{sech}\theta\right) + \operatorname{sech}\theta\tanh\theta = (\operatorname{sech}\theta)(\tanh\theta)(\ln\operatorname{sech}\theta)$.

#31. If $y = \arccos x - x\operatorname{arcsech} x$, find $\frac{dy}{dx}$. From the Sum Rule and the Product Rule, $\frac{dy}{dx} = \frac{d\arccos x}{dx} - \left(\frac{dx}{dx}\operatorname{arcsech} x + x\frac{d\operatorname{arcsech} x}{dx}\right)$. From formulae in the book, $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left(\operatorname{arcsech} x + x\frac{-1}{x\sqrt{1-x^2}}\right) = -\operatorname{arcsech} x$

#33. If $y = \operatorname{arccsch}\left(\left(\frac{1}{2}\right)^\theta\right)$ find $\frac{dy}{d\theta}$. By the Chain Rule and a formula in the book, $\frac{dy}{d\theta} =$

$$\frac{-\frac{d\left(\frac{1}{2}\right)^\theta}{d\theta}}{\left(\frac{1}{2}\right)^\theta\sqrt{1+\left(\left(\frac{1}{2}\right)^\theta\right)^2}} = \frac{\ln(\frac{1}{2})\left(\frac{1}{2}\right)^\theta}{\left(\frac{1}{2}\right)^\theta\sqrt{1+\left(\left(\frac{1}{2}\right)^\theta\right)^2}} = \frac{-\ln 2}{\sqrt{1+\left(\left(\frac{1}{2}\right)^\theta\right)^2}} = \frac{-2^\theta\ln 2}{\sqrt{2^{2\theta}+1}}$$

#39. Verify $\displaystyle\int x\operatorname{arccoth} x\,dx = \frac{x^2-1}{2}\operatorname{arccoth} x + \frac{x}{2} + C$. Compute $\frac{dy}{dx}$ where $y = \frac{x^2-1}{2}\operatorname{arccoth} x + \frac{x}{2}$. By the Sum Rule and the Product Rule, $\frac{dy}{dx} = \frac{d\frac{x^2-1}{2}}{dx}\operatorname{arccoth} x + \frac{x^2-1}{2}\frac{d\operatorname{arccoth} x}{dx} + \frac{1}{2}$. Plugging in the answers for the two derivatives (one by inspection and the other from the book) $\frac{dy}{dx} = x\operatorname{arccoth} x + \frac{x^2-1}{2}\frac{1}{1-x^2} + \frac{1}{2} = x\operatorname{arccoth} x + \frac{-1}{2} + \frac{1}{2}$. The integral formula is verified. One can *derive* formulae like this using Integration by Parts (§7.2).

#45. $\displaystyle\int\tanh\frac{x}{7}\ dx$. The $\frac{x}{7}$ is annoying so substitute $u = \frac{x}{7}$ ($du = \frac{dx}{7}$). $\displaystyle\int\tanh\frac{x}{7}\ dx = 7\int\tanh u\,du$. We concentrate on $\displaystyle\int\tanh u\,du$. By analogy with the ordinary trig. functions, try the substitution $w = \cosh u$. Then $dw = \sinh u\,du$ and $\displaystyle\int\tanh u\,du = \int\frac{\cosh u}{\sinh u}\,du = \int\frac{dw}{w} = \ln|w| + C = \ln|\cosh u| + C$. By definition, $\cosh u \geq 1$ so

$\int \tanh u\, du = \ln\cosh u + C$ and $\int \tanh\frac{x}{7}\ dx = 7\ln\cosh(\frac{x}{7}) + C$.

#49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\ dt}{\sqrt{t}}$. The $\sqrt{t}$ is annoying so substitute $u = \sqrt{t}$ ($du = \frac{dt}{2\sqrt{t}}$). $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\ dt}{\sqrt{t}} = 2\int \operatorname{sech} u \tanh u\, du$. From the book, $\int \operatorname{sech} u\tanh u\, du = -\operatorname{sech} u + C$, so $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\ dt}{\sqrt{t}} = -2\operatorname{sech}\sqrt{t}\ + C$.

#53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\ d\theta$. One could do this by Integration by Parts in analogy with $\int e^x\cos x\, dx$, but at this point in the book we have only substitution. If we recall that $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$, the substitution $u = e^{\theta}$ looks promising. Well $du = e^{\theta}d\theta$ so $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\ d\theta = 2\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{u+u^{-1}}{2}\ du = \int_{\frac{1}{4}}^{\frac{1}{2}} u + u^{-1}\ du = \frac{u^2}{2} + \ln|u|\Big|_{\frac{1}{4}}^{\frac{1}{2}} = \Big(\frac{(\frac{1}{2})^2}{2} + \ln\frac{1}{2}\Big) - \Big(\frac{(\frac{1}{4})^2}{2} + \ln\frac{1}{4}\Big) = \Big(\frac{1}{8} - \ln 2\Big) - \Big(\frac{1}{32} - \ln 4\Big) = \frac{3}{32} + 2\ln 2 - \ln 2 = \frac{3}{32} + \ln 2$

#77. **Part a).** With the verbiage removed, we are being asked to verify that $v = \sqrt{\frac{mg}{k}}\tanh\Big(\Big(\sqrt{\frac{gk}{m}}\Big)t\Big)$ satisfies the differential equation $m\frac{dv}{dt} = mg - kv^2$ with initial condition $v(0) = 0$. In this problem, m, g and k are constants. To simplify the writing (typing) let $a = \sqrt{\frac{gk}{m}}$ and check $v = \frac{g}{a}\tanh(at)$. $v(0) = \frac{g}{a}\tanh(a\cdot 0) = \frac{g}{a}\tanh 0 = \frac{g}{a}\frac{e^0-e^{-0}}{e^0+e^{-0}} = \frac{g}{a}\cdot 0 = 0$, which verifies the initial condition. Compute $\frac{dv}{dt} = \frac{g}{a}a\operatorname{sech}^2(at) = g\operatorname{sech}^2(at)$. We are being asked whether $m\frac{dv}{dt}$ is $mg - kv^2$. Well, $m\frac{dv}{dt} = mg\operatorname{sech}^2(at)$ and $mg - kv^2 = mg - k\big(\frac{g}{a}\tanh(ah)\big)^2 = mg - \frac{kg^2}{a^2}\tanh^2(at) = mg - mg\tanh^2(at)$. Since $\operatorname{sech}^2 u = 1 - \tanh^2 u$ the two quantities are equal.

Part b). We are being asked to compute $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\frac{g}{a}\tanh(at)$. Since $a > 0$, $\lim_{t\to\infty} e^{at} = \infty$ and $\lim_{t\to\infty} e^{-at} = 0$. Hence $\lim_{t\to\infty}\cosh(at) = \lim_{t\to\infty}\sinh(at) = \infty$ so are limit is of the form $\frac{\infty}{\infty}$ and hence a candidate for L'Hôpital's Rule. $\lim_{t\to\infty}\frac{g}{a}\tanh(at) = \lim_{t\to\infty}\frac{g}{a}\frac{\sinh(at)}{\cosh(at} = \frac{g}{a}\lim_{t\to\infty}\frac{a\cosh(at)}{a\sinh(at)} = \frac{g}{a}\lim_{t\to\infty}\frac{g}{a}\coth(at)$. This looks like no improvement so some other method must be employed. Recall $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ so we may rewrite $\tanh x = \frac{1-e^{-2x}}{1+e^{-2x}}$. Now our limit becomes $\lim_{t\to\infty}\frac{g}{a}\tanh(at) = \frac{g}{a}\lim_{t\to\infty}\frac{1-e^{-2atx}}{1+e^{-2at}} = \frac{g}{a}\frac{1-0}{1+0} = \frac{g}{a} = \sqrt{\frac{mg}{k}}$.

Part c). We are being asked to numerically evaluate $\sqrt{\frac{mg}{k}}$ when $mg = 160$ and $k = 0.005$ or to find $\sqrt{32000} = \sqrt{3.2\cdot 10^4} = \sqrt{3.2}\cdot 10^2 = 1.78885\cdot 10^2$.