§6.11

#3. Check $y = \frac{1}{x}\int_1^x \frac{e^t}{t}dt$ is a solution to $x^2y'+xy = e^x$. First compute $y' = \left(\frac{d\frac{1}{x}}{dx}\right)\int_1^x \frac{e^t}{t}dt + \frac{1}{x}\frac{d\int_1^x \frac{e^t}{t}dt}{dx}$. Now $\frac{d\frac{1}{x}}{dx} = \frac{-1}{x^2}$ and by the Fundamental Theorem of Calculus, $\frac{d\int_1^x \frac{e^t}{t}dt}{dx} = \frac{e^x}{x}$. Plugging in, we see $y' = \left(\frac{-1}{x^2}\right)\int_1^x \frac{e^t}{t}dt + \frac{1}{x}\frac{e^x}{x} = \left(\frac{-1}{x^2}\right)\int_1^x \frac{e^t}{t}dt + \frac{e^x}{x^2}$. Hence $x^2y' = -\int_1^x \frac{e^t}{t}dt + e^x$ and $xy = \int_1^x \frac{e^t}{t}dt$ so $x^2y' + xy = e^x$ as required.

#9. Solve $\frac{dy}{dx} = 2(x+y^2x)$. Since y^2 occurs, the equation is not linear. To separate, notice $x+y^2x = x(1+y^2)$: $\frac{dy}{1+y^2} = 2xdx$. Integrating, $\int \frac{dy}{1+y^2} = \int 2x\,dx$ or $\arctan y = x^2+C$ or $y = \tan(x^2+C)$.

#11. Solve $2\sqrt{xy}\frac{dy}{dx} = 1$, $x,y > 0$. Note $\sqrt{xy} = \sqrt{x}\sqrt{y}$ so the equation is separable and $2\sqrt{y}\,dy = \frac{dx}{\sqrt{x}}$. Integrating we see $2\int\sqrt{y}\,dy = \int\frac{dx}{\sqrt{x}}$ or $2\frac{y^{3/2}}{3/2} = \frac{x^{1/2}}{1/2} + C$ or $\frac{4}{3}y^{3/2} = 2x^{1/2} + C$. Hence $y^{3/2} = \frac{3}{2}x^{1/2} + C_1$ or $y = \sqrt[3]{(\frac{3}{2}x^{1/2} + C_1)^2}$

#17. Solve $xy' + 3y = \frac{\sin x}{x^2}$, $x > 0$. This is linear of the form $y' + Py = Q$ for $P = \frac{3}{x}$ and $Q = \frac{\sin x}{x^3}$. Find the integrating factor, $v = e^{\int P dx}$. $\int\frac{3dx}{x} = 3\ln|x|$ so $e^{\int Pdx} = e^{3\ln|x|} = |x^3|$. We may drop the $|\quad|$ or observe that since $x > 0$ in our problem, $|x^3| = x^3$. Hence $\frac{dv\cdot y}{dx} = v\left(\frac{dy}{dx}\right) + \left(\frac{dv}{dx}\right)y = x^3\left(\frac{dy}{dx}\right) + 3x^2y = x^2\left(\frac{dy}{dx} + 3y\right) = x^2\frac{\sin x}{x^2} = \sin x$. Hence $v\cdot y = -\cos x + C$ and so $y = \frac{-\cos x}{x^3} + \frac{C}{x^3}$. Even thought you were not asked, you can check directly that $y = \frac{-\cos x}{x^3} + \frac{C}{x^3}$ is also a solution if $x < 0$.

#23. Solve $e^{2x}y' + 2e^{2x}y = 2x$. Linear with $P = 2$, $Q = \frac{2x}{e^{2x}} = 2xe^{-2x}$. To find the integrating factor, compute $\int 2dx = 2x + C$ so we may take $v = e^{2x}$ as an integrating factor. Calculate $\frac{dv\cdot y}{dx} = vy' + \left(\frac{dv}{dx}\right)y = e^{2x}y' + 2e^{2x}y = 2x$. Hence $v\cdot y = x^2 + C$ so $y = \frac{x^2}{e^{2x}} + \frac{C}{e^{2x}} = x^2e^{-2x} + Ce^{-2x}$.

#29. Solve $(\sec^2\sqrt{x})\frac{dx}{dt} = \sqrt{x}$. This is NOT linear with the dependent variable x and independent variable t. It is separable: $\frac{(\sec^2 x)\,dx}{\sqrt{x}} = dt$. Integrate: $\int\frac{(\sec^2 x)\,dx}{\sqrt{x}} = \int dt$. The first integral can be done by a substitution: $w = \sqrt{x}$ so $dw = \frac{dx}{2\sqrt{x}}$ and $\int\frac{(\sec^2 x)\,dx}{\sqrt{x}} = 2\int\sec^2 w dw = 2\tan w + C = 2\tan(\sqrt{x}) + C$. Hence $2\tan(\sqrt{x}) = t + C$: $\sqrt{x} = \arctan(\frac{t}{2} + C_1)$ and $x = \left(\arctan(\frac{t}{2} + C_1)\right)^2$.

This equation IS linear with t as the dependent variable and x is the dependent variable since $\sec^2\sqrt{x} = \frac{dt}{dx}\sqrt{x}$ or $\frac{dt}{dx} = \frac{\sec^2\sqrt{x}}{\sqrt{x}}$. Hence $P = 0$ and $Q = \frac{\sec^2\sqrt{x}}{\sqrt{x}}$. When $P = 0$ you may take $v = 1$ but you just have to integrate Q. This is the same integral we had to do above.

#37. Solve $\theta\frac{dy}{d\theta} + y = \sin\theta$, $\theta > 0$ subject to the initial condition $y(\frac{\pi}{2}) = 1$. This equation is linear with $P = \frac{1}{\theta}$ and $Q = \frac{\sin\theta}{\theta}$. Ignore the initial condition for now and just solve the equation. The integrating factor is $v = e^{\int P d\theta} = e^{\ln|\theta|} = |\theta|$. We may take $v = \theta$. Compute $\frac{dv\cdot y}{d\theta} = v\frac{dy}{d\theta} + \left(\frac{dv}{d\theta}\right)y = \theta\frac{dy}{d\theta} + y = \sin\theta$ so $v \cdot y = (-\cos\theta) + C$ and $y = \frac{(C - \cos\theta)}{\theta}$. Now we turn to the initial condition, $y(\frac{\pi}{2}) = 1$. Recall $\cos(\pi/2) = 0$ so $1 = \frac{C-0}{\pi/2}$. Hence $\frac{\pi}{2} = C$. Hence $y = \frac{\frac{\pi}{2} - \cos\theta}{\theta} = \frac{-\cos\theta}{\theta} + \frac{\pi}{2\theta}$.

#47. We are told that the differential equation governing the height of water in the tank is $\frac{dy}{dt} = -k\sqrt{y}$. It is the province of physics or engineering to derive this equation, but we are given it. The constant k can be determined from experiment, but we are given that $k = \frac{1}{10}$. The time needed for the tank to drain is the solution to the equation $y(t) = 0$. (The tank is empty if and only if the height, y, is 0.) The fact that the level is 9 feet at the start means that $y(0) = 9$ if we begin measuring time when we open the valve.

Step 1. Solve the differential equation. The equation is separable and $\frac{dy}{\sqrt{y}} = -k\,dt$. Integrating $\int \frac{dy}{\sqrt{y}} = -k\int dt$ so $2y^{1/2} = -kt + C$ or $y^{1/2} = \frac{-kt}{2} + C_1$ and $y = \left(C_1 - \frac{kt}{2}\right)^2$.

Step 2. Find C_1 from the relation $y(0) = 9 = \left(C_1 - \frac{k\cdot 0}{2}\right)^2$ or $C_1^2 = 9$ so $C_1 = \pm 3$. To decide whether it is plus or minus, recall $y^{1/2} = \frac{-kt}{2} + C_1$. The square root function is always non–negative. Since $k = \frac{1}{10}$ and $t \geq 0$, $\frac{-kt}{2}$ is negative. Hence C_1 must be positive and so $C_1 = 3$.

Step 3. Solve the equation $y(t) = 0$ or $\left(3 - \frac{t}{20}\right)^2 = 0$ (plugging in $k = \frac{1}{10}$). Hence $3 - \frac{t}{20} = 0$ and hence $t = 60$. Since t is being measured in minutes, the time is 60 min. or one hour later.

Note how sensitive these sorts of equations are to the power of y. If $\frac{dy}{dt} = ky$ you get exponential growth or decay and y grows (or falls) faster than any power of t. If $\frac{dy}{dt} = ky^r$ for any $r \neq 1$ then y will be some algebraic function of t and grow (or falls) much more slowly.